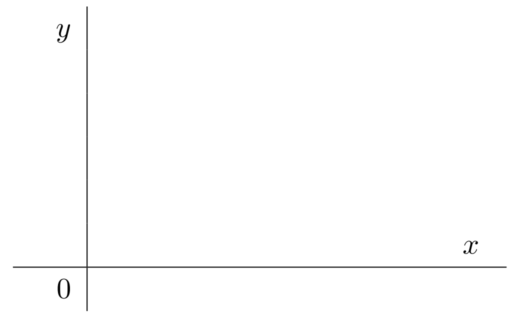
13.3: Double integrals over general regions

`All functions below are continuous on their domains.`

Let D be a bounded region enclosed in a rectangular region R. We define

$$F(x,y)=\begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not in } D. \end{cases}$$



If F is integrable over R, then we say F is *integrable* over D and we define **the double integral of** f **over** D by

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_R F(x,y)\,\mathrm{d}A$$

FACT: If $f(x,y)\geq 0$ and f is continuous on the region D then the volume V of the solid S that lies above D and under the graph of f, i.e.

$$S=\left\{(x,y,z)\in\mathbb{R}^3 \mid 0\leq z\leq f(x,y),(x,y)\in D\right\},$$

is

$$V=\iint_D f(x,y)\,\mathrm{d}A.$$

EXAMPLE 1. *Evaluate the integral*

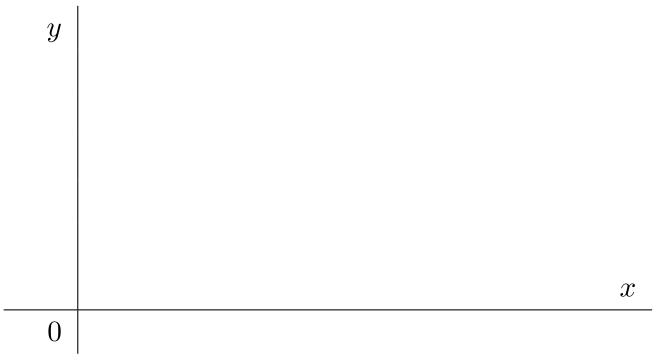
$$\iint_D \sqrt{16-x^2-y^2}\,\mathrm{d}A$$

where $D=\left\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\leq 16\right\}$ by identifying it as a volume of a solid.

Computation of double integral:

A plain region of **TYPE I**:

$$D = \{(x,y)|\ a \le x \le b, g_1(x) \le y \le g_2(x)\}.$$

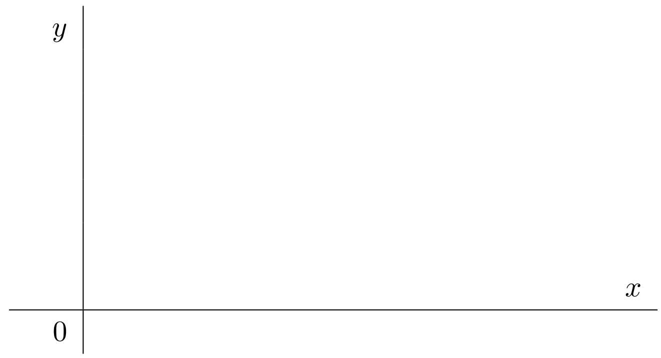


THEOREM 2. *If D is a region of type I such that $D = \{(x,y)|\ a \le x \le b, g_1(x) \le y \le g_2(x)\}$ then*

$$\iint_D f(x,y)\,\mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,\mathrm{d}y\mathrm{d}x.$$

A plain region of **TYPE II**:

$$D = \{(x,y)|\ c \le y \le d, h_1(y) \le x \le h_2(y)\}.$$



THEOREM 3. *If D is a region of type II s.t. $D = \{(x,y)|\ c \le y \le d, h_1(y) \le x \le h_2(y)\}$ then*

$$\iint_D f(x,y)\,\mathrm{d}A = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,\mathrm{d}x\mathrm{d}y.$$

EXAMPLE 4. *Evaluate* $I = \iint_D (x+y)\,\mathrm{d}A$, *where* D *is the region bounded by the lines* $x = 2, y = x$ *and the hyperbola* $xy = 1$.

EXAMPLE 5. *Find the volume of the solid bounded by the cylinder* $x^2 + y^2 = 1$ *and the planes* $x = 0, y = z, z = 0$ *in the first octant.*

EXAMPLE 6. *Evaluate the integral by reversing the order of integration:*

$$I = \int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\, \mathrm{d}y \mathrm{d}x.$$

EXAMPLE 7. *Sketch the region of integration and change the order of integration:*

$$\iint_D f(x,y)\,\mathrm{d}A = \int_0^1 \int_0^{\sqrt[3]{x^2}} f(x,y)\,\mathrm{d}y\mathrm{d}x + \int_1^2 \int_0^{1-\sqrt{1-(x-2)^2}} f(x,y)\,\mathrm{d}y\mathrm{d}x$$

EXAMPLE 8. *Evaluate the double integral*

$$I = \iint_D e^{\frac{x}{y}}\,\mathrm{d}A$$

where D is bounded by the lines

$$y = 1, y = 2, x = -y, x = y.$$

Properties of double integrals:

- If $D = D_1 \cup D_2$, where D_1 and D_2 do not overlap except perhaps their boundaries then

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_{D_1} f(x,y)\,\mathrm{d}A + \iint_{D_2} f(x,y)\,\mathrm{d}A.$$

- If α and β are real numbers then

$$\iint_D (\alpha f(x,y) + \beta g(x,y))\,\mathrm{d}A = \alpha \iint_D f(x,y)\,\mathrm{d}A + \beta \iint_D g(x,y)\,\mathrm{d}A.$$

- If we integrate the constant function $f(x,y) = 1$ over D, we get **area** of D:

$$\iint_D 1\,\mathrm{d}A = A(D).$$

EXAMPLE 9. *If* $D = \{(x,y) |\ x^2 + y^2 \le 25\}$ *then*

$$\iint_D \mathrm{d}A =$$