## 13.9-13.10: Part I

## Triple integrals in cylindrical coordinates

• Cylindrical coordinates:

 $P(x,y,z)\in\mathbb{R}^3$ 

In the cylindrical coordinates P is represented by the ordered triple  $(r, \theta, z)$ , where  $r, \theta$  are the polar coordinates of  $P_{xy}$  and z is the directed distance from the xy-plane to P:

x = y = z =

## where

$$r^2 = an \theta = z = z.$$

**REMARK 1.** The cylindrical coordinates

$$x = r \cos \theta$$
  

$$y = r \sin \theta$$
  

$$z = z$$
  

$$r \ge 0, \quad 0 \le \theta \le 2\pi$$

are useful in problems that involve symmetry about the z-axis.

EXAMPLE 2. Find an equation in cylindrical coordinates for the cone

$$z = \sqrt{x^2 + y^2}$$

THEOREM 3. Let f(x, y, z) be a continuous function over a solid  $E \subset \mathbb{R}^3$ . Let  $E^*$  be its image in cylindrical coordinates. Then

$$\iiint_E f(x, y, z) \, \mathrm{d}V = \iiint_{E^*} f(r \cos \theta, r \sin \theta, z) \, \mathrm{d}V^*,$$

where

$$\mathrm{d}V^* = r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\theta$$

EXAMPLE 4. The density at any point of the solid E,

$$E = \left\{ (x, y, z) : x^2 + y^2 \le 9, -1 \le z \le 4 \right\},\$$

equals to its distance from the axis of E. Find the mass of E.

REMARK 5. If E is a solid region of type I, i.e.

$$E = \{(x, y, z) | (x, y) \in D, \phi_1(x, y) \le z \le \phi_2(x, y)\},\$$

where D is the projection of E onto the xy-plane then, as we know,

$$\iiint_E f(x, y, z) \, \mathrm{d}V = \iint_D \left[ \int_{\phi_1(x, y)}^{\phi_2(x, y)} f(x, y, z) \, \mathrm{d}z \right] \mathrm{d}A.$$

Passing to cylindrical coordinates here we actually have to replace D by its image  $D^*$  in polar coordinates and dz dA by  $r dz dr d\theta$ .

EXAMPLE 6. Find the volume of the solid E bounded by the surfaces

$$y = x$$
,  $y = -x$ ,  $x^2 + y^2 = 5z$ ,  $z = 7$ 

so that  $y \ge 0$ .