13.9-13.10: Part I

Triple integrals in spherical coordinates

- **Spherical coordinates** of P is the ordered triple (ρ, θ, ϕ) where
  $|OP| = \rho$, $\rho \geq 0$, $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$.

We have

REMARK 1. The spherical coordinates

$$\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \\ z &= \rho \cos\phi \\ r &\geq 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi \end{aligned}$$

are especially useful in problems where there is *symmetry about the origin.*

Note that

$$x^2 + y^2 + z^2 =$$

EXAMPLE 2. *Find equation in spherical coordinates for the following surfaces.*

(a) $x^2 + y^2 + z^2 = 16$

(b) $z = \sqrt{x^2 + y^2}$

(c) $z = \sqrt{3x^2 + 3y^2}$

(d) $x = y$

•Triple integrals in spherical coordinates

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$
$$r \geq 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi$$

$$(x, y, z) \quad \rightarrow \quad (r, \theta, z) \quad \rightarrow \quad (\rho, \theta, \phi)$$

$$x = r\cos\theta \qquad z = \rho\cos\phi$$
$$y = r\sin\theta \qquad r = \rho\sin\phi$$
$$z = z \qquad \theta = \theta$$

$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z =$

THEOREM 3. *Let $f(x, y, z)$ be a continuous function over a solid $E \subset \mathbb{R}^3$. Let E^* be its image in spherical coordinates. Then*

$$\iiint_E f(x, y, z)\,\mathrm{d}V = \iiint_{E^*} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

EXAMPLE 4. *Evaluate* $I = \iiint_E e^{\sqrt{(x^2+y^2+z^2)^3}}\,\mathrm{d}V$ *where* $E = \{(x,y,z) : 9 \leq x^2+y^2+z^2 \leq 16\}$.

EXAMPLE 5. *Write the integral* $\iiint_E f(x,y,z)\,\mathrm{d}V$ *in spherical coordinates where*

(a) $E = \{(x,y,z) : x^2+y^2+z^2 \leq 1, y \geq 0, z \geq 0\}$.

(b) E *is the icecream cone-shaped solid, which is cut from the sphere of radius* 5 *by the cone* $\phi = \pi/6$.

EXAMPLE 6. *Evaluate the integral by changing to spherical coordinates:*

$$I = \int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y.$$