

## 14.2: Line Integrals

**Line integrals on plane:** Let  $C$  be a plane curve with parametric equations:

$$x = x(t), y = y(t), \quad a \leq t \leq b,$$

or we can write the parametrization of the curve as a vector function:

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle, \quad a \leq t \leq b.$$

**DEFINITION 1.** *The line integral of  $f(x, y)$  with respect to arc length, or the **line integral of  $f$  along  $C$  is***

$$\int_C f(x, y) \, ds$$

Recall that the *arc length* of a curve given by parametric equations  $x = x(t), y = y(t), \quad a \leq t \leq b$  can be found as

$$L = \int_a^b ds,$$

where

$$ds = \sqrt{(x'(t))^2 + (y'(t))^2} \, dt.$$

The line integral is then

$$\int_C f(x, y) \, ds =$$

If we use the vector form of the parametrization we can simplify the notation up noticing that

$$\mathbf{r}'(t) = \langle x'(t), y'(t) \rangle$$

and then

$$ds = \sqrt{(x'(t))^2 + (y'(t))^2} \, dt =$$

Using this notation the line integral becomes,

$$\int_C f(x, y) \, ds = \int_a^b f(x(t), y(t)) |\mathbf{r}'(t)| \, dt.$$

**REMARK 2.** The value of the line integral does not depend on the parametrization of the curve, provided that *the curve is traversed exactly once as  $t$  increases from  $a$  to  $b$ .*

Let us emphasize that  $ds = |r'(t)| \, dt = \sqrt{[x'(t)]^2 + [y'(t)]^2} \, dt.$

EXAMPLE 3. Evaluate the line integral  $\int_C y \, ds$ , where  $C : x = t^3, y = t^2, 0 \leq t \leq 1$ .

**Line integrals in space:** Let  $C$  be a space curve with parametric equations:

$$x = x(t), y = y(t), z = z(t), \quad a \leq t \leq b,$$

or

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a \leq t \leq b.$$

The **line integral of  $f$  along  $C$**  is

$$\int_C f(x, y, z) \, ds = \int_a^b f(x(t), y(t), z(t)) |r'(t)| \, dt.$$

Here

$$ds = |r'(t)| \, dt = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} \, dt.$$

EXAMPLE 4. Evaluate the line integral  $\int_C (x + y + z) \, ds$ , where  $C$  is the line segment joining  $(-1, 1, 2)$  and  $(2, 3, 1)$ .

**Physical interpretation of a line integral:** Let  $\rho(x, y, z)$  represents the linear density at a point  $(x, y, z)$  of a thin wire shaped like a curve  $C$ . Then the **mass**  $m$  of the wire is:

$$m = \int_C \rho(x, y, z) \, ds.$$

**EXAMPLE 5.** A thin wire with the linear density  $\rho(x, y) = x^2 + 2y^2$  takes the shape of the curve  $C$  which consists of the arc of the circle  $x^2 + y^2 = 1$  from  $(1, 0)$  to  $(0, 1)$ . Find the mass of the wire.

**Line integrals with respect to  $x, y$ , and  $z$ .** Let  $C$  be a space curve with parametric equations:

$$x = x(t), y = y(t), z = z(t), \quad a \leq t \leq b,$$

The **line integral of  $f$  with respect to  $x$**  is,

$$\int_C f(x, y, z) \, dx = \int_a^b f(x(t), y(t), z(t))x'(t) \, dt.$$

The **line integral of  $f$  with respect to  $y$**  is,

$$\int_C f(x, y, z) \, dy = \int_a^b f(x(t), y(t), z(t))y'(t) \, dt.$$

The **line integral of  $f$  with respect to  $z$**  is,

$$\int_C f(x, y, z) \, dz =$$

These two integral often appear together by the following notation:

$$\int_C P \, dx + Q \, dy + R \, dz$$

or

$$\int_C P \, dx + Q \, dy.$$

EXAMPLE 6. *Compute*

$$I = \int_C -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy,$$

where  $C$  is the circle  $x^2 + y^2 = 1$  oriented in the counterclockwise direction.

**Line integrals of vector fields.**

*PROBLEM:* Given a continuous force field,

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k},$$

such as a gravitational field. Find the work done by the force  $\mathbf{F}$  in moving a particle along a curve

$$C : \quad \mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a \leq t \leq b.$$

*SOLUTION:*

DEFINITION 7. Let  $\mathbf{F}$  be a continuous vector field defined on a curve  $C$  given by a vector function  $\mathbf{r}(t)$ ,  $a \leq t \leq b$ . Then the **line integral of  $\mathbf{F}$  along  $C$**  is

$$\int_C \mathbf{F} \cdot d\mathbf{r}(t) = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt.$$

REMARK 8. Note that this integral depends on the curve orientation:

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r}(t) = - \int_C \mathbf{F} \cdot d\mathbf{r}(t)$$

EXAMPLE 9. Find the work done by the force field  $\mathbf{F}(x, y, z) = \langle xy, yz, xz \rangle$  in moving a particle along the curve  $C : \mathbf{r}(t) = \langle t, t^2, t^3 \rangle, 0 \leq t \leq 1$ .

**Relationship between line integrals of vector fields and line integrals with respect to  $x, y$ , and  $z$ .**

$$\int_C \mathbf{F} \cdot d\mathbf{r}(t) =$$