## 14.5: Curl and Divergence

Introduce the vector differential operator  $\nabla$  as

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}.$$

If  $\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$  is a vector field on  $\mathbf{R}^3$  and the partial derivatives of P,Q,R all exist, then the **curl** of  $\mathbf{F}$  is the vector field on  $\mathbf{R}^3$  defined by

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} =$$

EXAMPLE 1. Find the curl of the vector field

$$\mathbf{F}(x,y,z) = \langle xy, x^2, yz \rangle.$$

Question What is the curl of a two-dimensional vector field

$$\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$$
?

Answer:

CONCLUSION: Green's Theorem in vector form:

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} =$$

THEOREM 2. If a function f(x, y, z) has continuous partial derivatives of second order then

$$\operatorname{curl}(\nabla f) = 0.$$

Proof:

COROLLARY 3. If  $\mathbf{F}$  is conservative, then  $\operatorname{curl} \mathbf{F} = \mathbf{0}$ .

The proof of the Theorem below requires Stokes' Theorem (Section 14.8).

THEOREM 4. If  $\mathbf{F}$  is a vector field defined on  $\mathbf{R}^3$  whose component functions have continuous partial derivatives and  $\mathrm{curl}\mathbf{F} = 0$ , then  $\mathbf{F}$  is a conservative vector field.

EXAMPLE 5. Let  $\mathbf{F}(x, y, z) = \langle x^9, y^9, z^9 \rangle$ .

(a) Show that **F** is conservative.

(b) Find a function f s.t.  $\nabla f = \mathbf{F}$ .

(c) Evaluate  $\int_{(1,0,1)}^{(-1,-1,-1)} \mathbf{F} \cdot d\mathbf{r}$ .

If  $\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$  is a vector field on  $\mathbf{R}^3$  and the partial derivatives  $P_x, Q_y, R_z$  exist, then the **divergence of F** is the *scalar field* on defined by

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} =$$

EXAMPLE 6. Find the divergence of the vector field

$$\mathbf{F}(x, y, z) = \langle \sin(xyz), x^2, yz \rangle.$$

THEOREM 7. If the components of a vector field  $\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$  has continuous partial derivatives of second order then

div curl 
$$\mathbf{F} = 0$$
.

Proof.

EXAMPLE 8. Is there a vector field **G** on  $\mathbb{R}^3$  s.t. curl  $G = \langle yz, xyz, zy \rangle$ ?