14.6: Parametric surfaces and their areas

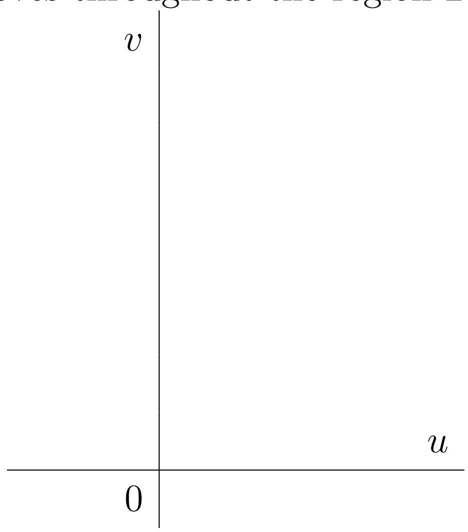
Consider a continuous vector valued function of two variables

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}, \quad (u,v) \in D.$$

Parametric surface:

$$S: \; x = x(u,v), \quad y = y(u,v), \quad z = z(u,v), \quad (u,v) \in D.$$

In other words, the surface S is traces out by the position vector $\mathbf{r}(u,v)$ as (u,v) moves throughout the region D.



EXAMPLE 1. *Determine the surface given by the parametric representation*

$$\mathbf{r}(u,v) = \langle u, u\cos v, u\sin v \rangle, \quad 1 \le u \le 5, \quad 0 \le v \le 2\pi$$

EXAMPLE 2. *Give parametric or vector representations for each of the following surfaces:*

(a) *the cylinder:* $x^2 + y^2 = 9, \quad 1 \leq z \leq 5.$

(b) *the upper half-sphere:* $z = \sqrt{100 - x^2 - y^2}.$

(c) *elliptic paraboloid:* $x = 3y^2 + z^2 + 1.$

(d) *elliptic paraboloid* $y = x^2 + 4z^2$

CONCLUSION: To parametrize surface we may use polar, cylindrical or spherical coordinates, or

- $z = f(x, y) \longrightarrow \mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}$
- $y = f(x, z) \longrightarrow \mathbf{r}(x, z) = x\mathbf{i} + f(x, z)\mathbf{j} + z\mathbf{k}$
- $x = f(y, z) \longrightarrow \mathbf{r}(y, z) = f(y, z)\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

• **Tangent planes**:

PROBLEM: Find a normal vector to the tangent plane to a parametric surface S *given by a vector function* $\mathbf{r}(u, v)$ *at a point* P_0 *with position vector* $\mathbf{r}(u_0, v_0)$, i.e. $P_0(x(u_0, v_0), y(u_0, v_0), z(u_0, v_0))$

Solution:

The **normal vector**

$$\mathbf{N} = \mathbf{N}(u, v) =$$

If a normal vector is not $\mathbf{0}$ then the surface S is called **smooth** (it has no "corner").

EXAMPLE 3. *Find the tangent plane to the surface with parametric equations* $x = uv + 1, y = ue^v, z = ve^u$ *at the point* $(1, 0, 0)$.

Special Case: a surface S given by a *graph* $z = f(x, y)$. Then one can choose the following parametrization of S:

$$\mathbf{r}(x, y) =$$

and the then the normal vector is

$$\mathbf{N} =$$

- **Surface Area:**

Consider a smooth surface S given by

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}, \quad (u, v) \in D,$$

then

$$\mathrm{d}S = |N(u, v)|\mathrm{d}u\mathrm{d}v =$$

and the **surface area**

$$A(S) = \iint_S \mathrm{d}S = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, \mathrm{d}A.$$

REMARK 4. *Special Case*: a surface S given by a *graph* $z = f(x, y)$ we have

$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}$$

and

$$\mathrm{d}S = |\mathbf{N}(x, y)| \, \mathrm{d}A =$$

EXAMPLE 5. *Find the surface area of the surface*

$$S: \quad x = uv, \quad y = u + v, \quad z = u - v, \qquad u^2 + v^2 \leq 1.$$

EXAMPLE 6. *Find the surface area of the part paraboloid* $z = x^2 + y^2$ *between two planes:* $z = 0$ *and* $z = 4$.