4. Sets

4.1. The language of sets

• **Set Terminology and Notation**

Set is a well-defined collection of objects. **Elements** are objects or members of the set.

Describing a Set

- **Roster notation:**
  $A = \{a, b, c, d, e\}$ Read: Set A with elements a, b, c, d, e.
- **Indicating a pattern:**
  $B = \{a, b, c, ..., z\}$ Read: Set B with elements being the letters of the alphabet.

If a is an element of a set A, we write $a \in A$ that read "a belongs to A." However, if a does not belong to A, we write $a \notin A$.

Set-builder notation (a more precise way of describing a set)

NOTATION 1. *Let $P(x)$ be a predicate. Then the notation*

$$A = \{x | P(x)\} \quad \text{or} \quad A = \{x : P(x)\}$$

denotes the set A of all elements x such that $P(x)$ is a true statement. In symbols,

$$\forall x, \ x \in A \ \Leftrightarrow P(x).$$

When D is a set containing the set A, the notation

$$A = \{x \in D | \ P(x)\} = \{x | \ x \in D \quad \wedge \quad P(x)\}$$

denotes the set A of all elements in D such that $P(x)$ is a true statement. In symbols,

$$\forall x \in D, \ x \in A \ \Leftrightarrow P(x).$$

EXAMPLE 2. *Use set-builder notation to describe the following sets in two different ways.*

a) $\mathbf{O}$

b) $5\mathbf{Z}$

c) $\mathbf{N}$

d) $\mathbf{Q}$

e) *Set of all integers of the form $4n + 2$.*

f) *Set of all positive integers less than 2019.*

EXAMPLE 3. *For each of the following sets use symbols to fill in the blanks:*

- $A = \{n | n \in \mathbb{E} \quad and \quad |n| > 12\}$
  1. $x \in A \Leftrightarrow$ ____________________________
  2. $16 \in A$ *because* ____________________________
  3. $4 \notin A$ *because* ____________________________
  4. $7 \notin A$ *because* ____________________________
- $B = \left\{x \in \mathbb{R} | x^2 - 4 = 0\right\}$
  1. $x \in B \Leftrightarrow$ ____________________________
  2. $16 \notin B$ *because* ____________________________
  3. $2 \in B$ *because* ____________________________
- $C = \{3t + 1 | t \in \mathbb{Z}\}$
  1. $x \in C \Leftrightarrow$ ____________________________
  2. $16 \in C$ *because* ____________________________
  3. $5 \notin C$ *because* ____________________________

Interval notation:

Intervals

NOTATION 4. • *bounded intervals:*

1. *closed interval* $[a, b] = \{x \in \mathbb{R} |\ a \le x \le b\}$
2. *open interval* $(a, b) = \{x \in \mathbb{R} |\ a < x < b\}$
3. *half-open,half-closed interval* $(a, b] = \{x \in \mathbb{R} |\ a < x \le b\}$
4. *half-closed,half-open interval* $[a, b) = \{x \in \mathbb{R} |\ a \le x < b\}$

- *unbounded intervals:*

5. $[a, \infty) = \{x \in \mathbb{R} |\ a \le x\}$
6. $(a, \infty) = \{x \in \mathbb{R} |\ a < x\}$
7. $(-\infty, a] = \{x \in \mathbb{R} | x \le a\}$
8. $(-\infty, a) = \{x \in \mathbb{R} |\ x < a\}$
9. $(-\infty, \infty) = \mathbb{R}$

EXAMPLE 5. *Represent the following sets in interval notation when it is possible.*

a) $\{x \in \mathbf{R} |\ (x \ge 0) \wedge (x \in \mathbf{Z})\} =$

b) $\{x \in \mathbf{Z} |\ 3 \le x < 10\} =$

c) $\{x \in \mathbf{R} |\ -2019 \le x \le 2020\} =$

Subsets

- Two sets, A and B, are **equal**, written $A = B$, if and only if they have exactly the same elements. (NOTE: they do not have to be in the same order!).

  For example,

  $$\{a, b, c\} \quad \{c, a, b\} \quad \{a, b, c, b\}$$

  `Question:` How to show that two sets are not equal?

- If every element in set A is also an element in set B, then A is a subset of B, written $A \subseteq B$.

  Note that $A \subseteq A$. In symbols: $A \subseteq B \Leftrightarrow (\forall x \in U, (x \in A \qquad x \in B))$

- If $A \subseteq B$, but $A \neq B$, then A is a **proper** subset of B, written $A \subset B$.

  $$A \subseteq B \Leftrightarrow (A \subset B \quad \vee \quad A = B)$$

  and

  $$A \subset B \Leftrightarrow (A \subseteq B \quad \wedge \quad A \neq B)$$

- The **empty set** is the set that doesn't have any elements, denoted by $\emptyset$ or $\{\}$.

- The **universal set** is the set that contains all of the elements for a problem, denoted by U.

EXAMPLE 6. *Let* $A = \{n \in \mathbf{Z} | n \textit{ is even}\}$, $B = \{n \in \mathbf{Z} | n^2 \textit{ is even}\}$, *and* $C = \{n^2 | n \textit{ is even}\}$. *Prove or disprove the following:*

(a) $A = B$

(b) $B = C$

infinite sets $\mathbb{R}, \mathbb{Z}, \mathbb{Q}, [1, 3), \{2^n | n \in \mathbb{N}\}$

finite sets $\{\Delta, \Box\}, \{2^n | n \in \{3, 4, 5\}\}$

cardinality of a finite set A, $|A|$

$$|\emptyset| = \qquad , \quad \left|\{x \in \mathbb{R} | x^4 = 1\}\right| =$$

EXAMPLE 7. *Let A and B be two sets.*

(a) ***TRUE/FALSE*** *If $A = B$, then $|A| = |B|$.*

(b) ***TRUE/FALSE*** *If $|A| = |B|$, then $A = B$.*

EXAMPLE 8. *Let $A = \{n \in \mathbb{Z} | n = 3t - 2 \text{ for some } t \in \mathbb{Z}\}$ and $B = \{n \in \mathbb{Z} | n = 3t + 1 \text{ for some } t \in \mathbb{Z}\}$. Prove that $A = B$.*

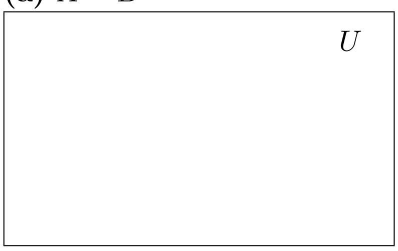
4.2 Operations on sets

VENN DIAGRAMS

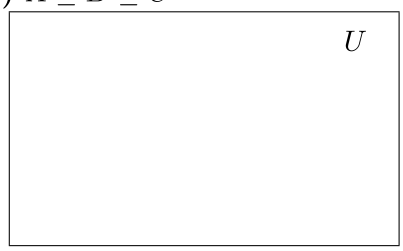
- a visual representation of sets (the universal set U is represented by a rectangle, and subsets of U are represented by regions lying inside the rectangle).

EXAMPLE 9. *Use Venn diagrams to illustrate the following statements:*

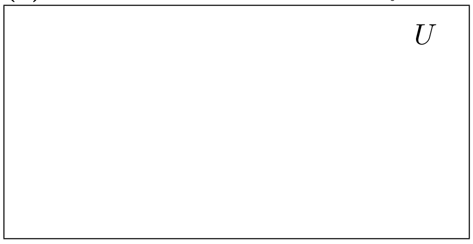
(a) $A = B$

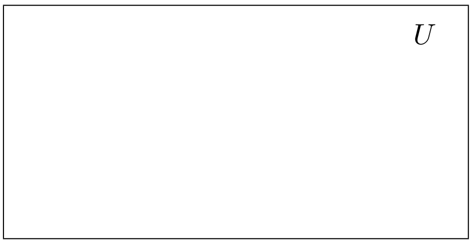


(b) $A \subseteq B \subseteq C$



(c) *A and B are not subsets of each other.*



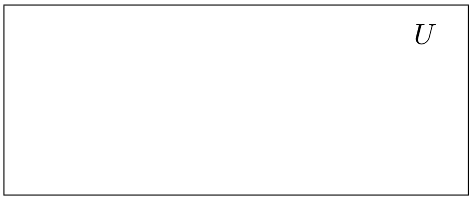


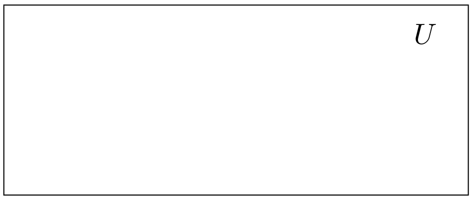
DEFINITION 10. *Let A and B be sets in a universal set U. The* **union** *of A and B, written $A \cup B$, is the set of all elements that belong to either A or B or both. Symbolically:*

$$A \cup B = \{x \in U | x \in A \vee x \in B\}.$$

DEFINITION 11. *Let A and B be sets in a universal set U. The* ***intersection*** *of A and B, written $A \cap B$, is the set of all elements in common with A and B. Symbolically:*

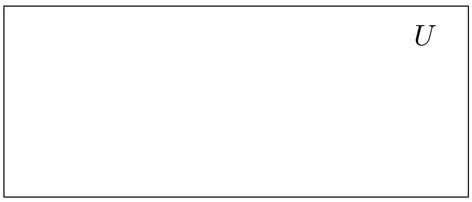
$$A \cap B = \{x \in U | x \in A \ \wedge \ x \in B\}.$$



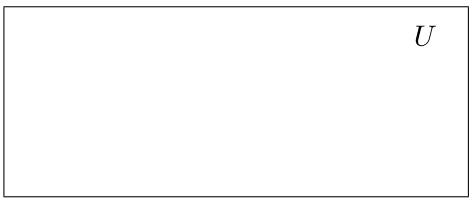


DEFINITION 12. *Let A and B be sets. The* **complement of** A **in** B *denoted $B - A$, is*

$$B - A = \{x \in U | x \in B \ \wedge \ x \notin A\}$$



REMARK 13. For convenience, if U is a universal set and A is a subset in U, we will write $U - A = \bar{A}$, called simply the **complement** of A.



EXAMPLE 14. *Let* $U = \{0, 1, 2, \dots, 9, 10\}$ *be a universal set,* $A = \{0, 2, 4, 6, 8, 10\}$*, and* $B = \{1, 3, 5, 7, 9\}$*. Find the following sets:*

(a) $\overline{A} \cup \overline{B}$

(b) $(\overline{A \cap B}) \cap (\overline{A \cup B})$

set notation	$=$	$\subset, \subseteq$	$\cup$	$\cap$	$\bar{\square}$	$\emptyset$	U
logical symbol							

Power set

DEFINITION 15. *Let* A *be a set. The power set of* A*, written* $P(A)$*, is the following set*

$$P(A) = \{X \mid X \subseteq A\}.$$

In other words, $P(A)$ is the set of all subsets of A (including $\emptyset$ and A).

EXAMPLE 16. *Find* $P(\{x, y\})$ *and fill in the blanks.*

$$P(\{x, y\}) =$$

(a) $\{x\}$ __ $\{x, y\}$ **(b)** $\{x\}$ __ $P(\{x, y\})$ **(c)** $\{\{x\}\}$ __ $P(\{x, y\})$ **(d)** $\emptyset$ __ $\{x, y\}$

(e) $\emptyset$ __ $P(\{x, y\})$ **(f)** $\emptyset$ __ $P(\{x, y\})$ **(g)** $\{\emptyset\}$ __ $P(\{x, y\})$

EXAMPLE 17. *Let* $A = \{-1, 0, 1\}$.

1. *Find all elements of power set of* A.

2. *Find* $|P(A)|$ *(the number of subsets of* A*) and* $|P(P(A))|$ *(the number of subsets of* $P(A)$*).*

3. *Write 3 subsets of* A *and* 5 *subsets of* $P(A)$.

4. *What are* $|P(A)|$ *and* $|P(P(A))|$ *for an arbitrary set* A*?*

EXAMPLE 18. *Find*

(a) $P(\{\Delta\})$

(b) $P(\emptyset)$

(c) $P(P(\emptyset))$

(d) $P(\{\Delta, \Box\})$

(e) $P(\{\emptyset, \{\emptyset\}\})$

Cartesian Product

DEFINITION 19. *Let A and B be sets. The* **Cartesian product** *of A and B, written $A \times B$, is the following set:*

$$A \times B = \{(a,b)|\ a \in A \wedge b \in B\}.$$

Informally, $A \times B$ is the set of **ordered** pairs of objects.

EXAMPLE 20. *Given $A = \{0,1\}$ and $B = \{4,5,6\}$.*

(a) *Does the pair $(6,1)$ belong to $A \times B$?*

(b) *List the elements of $A \times B$.*

(c) *What is the cardinality of $A \times B$?*

(d) *List the elements of $A \times A \times A$ and $(A \times A) \times A$.*

(e) *Does the triple $(1,6,4)$ belong to $A \times B \times B$?*

(f) *Describe the following sets $R \times R$, $R \times R \times R$.*

Proofs Involving Sets

Proving set properties

Use the following tautologies:

- $x \in A \cap B \Leftrightarrow (x \in A \wedge x \in B)$
- $x \in A \cup B \Leftrightarrow (x \in A \vee x \in B)$
- $x \in \bar{A} \Leftrightarrow x \notin A$
- $x \in A - B \Leftrightarrow (x \in A \wedge x \notin B)$
- $A = B \Leftrightarrow (x \in A \Leftrightarrow x \in B)$
- $A \subseteq B \Leftrightarrow (x \in A \Rightarrow x \in B)$
- $(x, y) \in A \times B \Leftrightarrow (x \in A \wedge y \in B)$

Methods:

- To prove $A \subseteq B$ it is sufficient to prove $x \in A \Rightarrow x \in B$.
- To prove $A = B$ it is sufficient to prove $x \in A \Leftrightarrow x \in B$.
- To prove $A = B$ it is sufficient to prove $A \subseteq B$ and $B \subseteq A$.
- To show that $A = \emptyset$ it is sufficient to show that $x \in A$ implies a false statement.

Fundamental properties of sets

THEOREM 21. *The following statements are true for all sets A, B, and C.*

1. $A \cup B = B \cup A$ (`commutative`)
2. $A \cap B = B \cap A$ (`commutative`)
3. $(A \cup B) \cup C = A \cup (B \cup C)$ (`associative`)
4. $(A \cap B) \cap C = A \cap (B \cap C)$ (`associative`)
5. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (`distributive`)
6. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (`distributive`)

   `DeMorgan's Laws:` *If A and B are the sets contained in some universal set U then*

7. $\overline{A \cup B} = \bar{A} \cap \bar{B}$.
8. $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

THEOREM 22. *Let A and B be a subsets of a universal set U. Then*

1. $\overline{\overline{A}} = A$.

2. $\overline{\emptyset} = U$.

3. $\overline{U} = \emptyset$

4. $A \subseteq A \cup B$.

5. $A \cap B \subseteq A$.

6. *The empty set is a subset of every set. (Namely, for every set A, $\emptyset \subseteq A$. If $A \neq \emptyset$, then $\emptyset \subset A$.).*

7. $A \cup \emptyset = A$.

8. $A \cap \emptyset = \emptyset$.

9. $A \cap A = A \cup A = A$

EXAMPLE 23. *Let A and B be subsets of a universal set U. Show that $(A - B) \cap B = \emptyset$.*

PROPOSITION 24. *Let A and B be subsets of a universal set U. Then*

$$A - B = A \cap \bar{B}.$$

EXAMPLE 25. *Let A, B and C be sets. Prove that*

$$A - (B \cup C) = (A - B) \cap (A - C)$$

EXAMPLE 26. *For the sets A, B and C prove that*

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

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PROPOSITION 27. *Let $A, B,$ and C be sets, and suppose $A \subseteq B$ and $B \subseteq C$. Then $A \subseteq C$.*

EXAMPLE 28. *Let A, B, C and D be sets. If $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$.*

EXAMPLE 29. *Prove the following statement. Let A and B be subsets of a universal set U. Then $A \subseteq B \Leftrightarrow A \cup B = B$.*

EXAMPLE 30. *Let A and B be subsets of a universal set U. Prove that*

$$A = A - B \Leftrightarrow A \cap B = \emptyset.$$

4.3 Arbitrary unions and intersections

DEFINITION 31. *Let I be a set. An* ***indexed collection of sets*** $\{A_\alpha\}_{\alpha \in I}$ *represents a collection of sets such that for every $\alpha \in I$, there is a corresponding set A_α. In this case we call I the* **indexed set.**

Collection of sets	Indexed set	Shortened notation
$A_0, A_1, A_2, A_3, \ldots, A_{2016}$		
B_3, B_6, B_9, B_{77}		
$C_5, C_{10}, C_{15}, \ldots, C_{2015}$		

- **Union and Intersection**

EXAMPLE 32. *Complete the following*

(a) $x \in \bigcup_{\alpha \in I} A_\alpha \Leftrightarrow \exists \alpha \in I \ni x \in A_\alpha$

$x \notin \bigcup_{\alpha \in I} A_\alpha \Leftrightarrow$

(b) $x \in \bigcap_{\alpha \in I} A_\alpha \Leftrightarrow \forall \alpha \in I, x \in A_\alpha$

$x \notin \bigcap_{\alpha \in I} A_\alpha \Leftrightarrow$

EXAMPLE 33. *Given* $B_i = \{i, i+1\}$ *for* $i = 1, 2, \ldots, 10$. *Determine the following*

(a) $\bigcap_{i=1}^{10} B_i$

(b) $B_i \cap B_{i+1}$

(c) $\bigcap_{i=k}^{k+1} B_i$ *where* $1 \le k < 10$.

(d) $\bigcup_{i=j}^{k} B_i$ *where* $1 \le j < k \le 10$.

EXAMPLE 34. $A_n = \left\{x \in \mathbf{R} \mid -\frac{1}{n} \le x \le \frac{1}{n}\right\}, \quad n \in \mathbf{Z}^+$. *Find* $\bigcup_{n \in \mathbf{Z}^+} A_n$ *and* $\bigcap_{n \in \mathbf{Z}^+} A_n$.