5 FUNCTIONS

5.1 Definition and Basic Properties

DEFINITION 1. *Let X and Y be nonempty sets. A* **function** *f from the set X to the set Y is a correspondence that assigns to each element x in the set X one and only one element y in the set Y, which is denoted by $f(x)$.*

We call X the **domain** *of f and Y the* **codomain** *of f.*

If $x \in X$ and $y \in Y$ are such that $y = f(x)$, then y is called the **value** of f at x, or the **image** of x under f. We may also say that f **maps** x to y.

Using diagram

DEFINITION 2. *Two functions f and g are* **equal** *if they have the same domain and the same codomain and if $f(x) = g(x)$ for all x in domain.*

DEFINITION 3. *The* **graph** *of $f : X \to Y$ is the set*

$$G_f = \{(x, y) \in X \times Y | y = f(x)\} .$$

- We can determine a function from its domain, codomain, and graph.
- We can describe a function by formula, by listing its values, or by words.

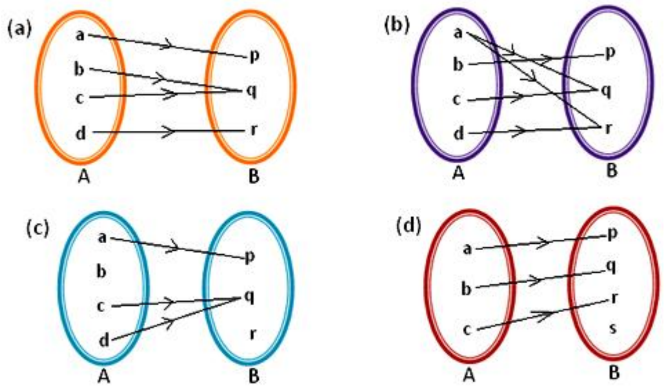
EXAMPLE 4. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 + 3$, $g : \mathbb{R} \to [0, \infty)$ be defined by $g(x) = x^2 + 3$, and $h : \{-1, 0, 1\} \to \mathbb{R}$ be defined by $h(x) = x^2 + 3$.*

(a) *Determine whether $f = g$.*

(b) *Determine whether $f = h$.*

(c) *Find the graphs of f, g, and h.*

EXAMPLE 5. *Decide if the following diagrams define functions from A to B.*



EXAMPLE 6. *Let* $X = \{2, 4, 6\}$ *and* $Y = \{a, b, c, d\}$. *Let* f *be a function defined by* $f(2) = b, f(4) = a, f(6) = d$ *and let* g *be a function from* X *to* Y *defined by its graph* $G_g = \{(2, c), (4, c), (6, c)\}$ *Find the following.*

(a) *The image of* 2 *under* f.

(b) *The image of* 6 *under* g.

(c) *The preimage of* d *under* f.

(d) *The preimage of* c *under* g.

(e) *The preimage of* d *under* g.

(f) *The codomain of* g.

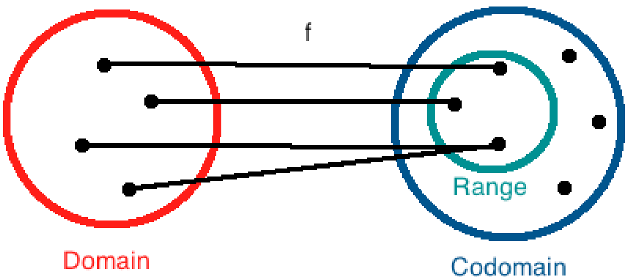
(g) G_f

Range (or Image) of a Function

DEFINITION 7. *Let* $f : X \to Y$ *be a function. The* **range** *of* f *(also called the* ***image*** *of* f*) is the set*

$$\{y \in Y | y = f(x) \quad for \quad some \quad x \in X\}.$$

We denote the range (or image) of the function f by $\text{ran} f$ (or $\text{Im} f$).



EXAMPLE 8. *Let* $f : X \to Y$ *be a function. Using symbols complete the following*

- $\text{ran} f \subseteq$ ____________
- $\forall y \in Y, y \in \text{ran} f \;\; \Leftrightarrow$ __________________________________
- $y \notin \text{ran} f \;\; \Leftrightarrow$ __________________________________

EXAMPLE 9. *$f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = \cos x$. Find* $\text{ran} f$.

EXAMPLE 10. *Let $f : [\frac{1}{3}, \infty) \to \mathbb{R}$ be defined by $f(x) = \sqrt{3x - 1}$ and $S = \{y \in \mathbb{R} | \ y \geq 0\}$. Prove that* $\text{ran} f = S$.

5.2 Composition of Functions

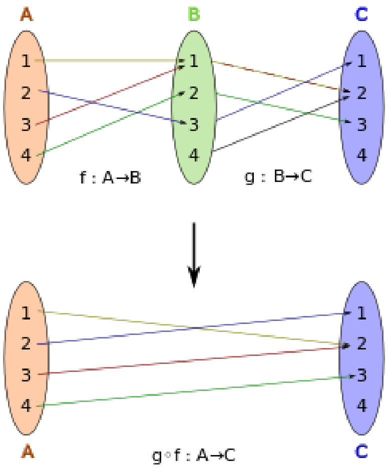
DEFINITION 11. *Let A, B, and C be nonempty sets, and let $f : A \to B$, $g : B \to C$. We define a function*

$$g \circ f : A \to C,$$

called the **composition** *of f and g, by*

$$(g \circ f)(a) =$$

Using diagram



EXAMPLE 12. *Let* $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$, $C = \{r, s, t, u, v\}$ *and define the functions* $f : A \to B$, $g : B \to C$ *by their graphs:*

$$G_f = \{(1, b), (2, d), (3, a), (4, a)\}, \qquad G_g = \{(a, u), (b, r), (c, r), (d, s)\}.$$

Find $g \circ f$*. What about* $f \circ g$*?*

EXAMPLE 13. *Let* $f, g : \mathbb{R} \to \mathbb{R}$ *be defined by* $f(x) = e^x$ *and* $g(x) = x \sin x$*. Find* $f \circ g$ *and* $g \circ f$*.*

PROPOSITION 14. *Let $f : A \to B$, $g : B \to C$, and $h : C \to D$. Then*

$$(h \circ g) \circ f = h \circ (g \circ f),$$

i.e. composition of functions is associative.

Proof.

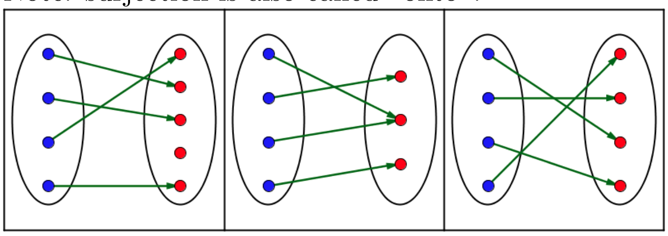
Section 5.3 Surjective (or onto) and Injective (or one-to-one) Functions

Surjective functions ("onto")

DEFINITION 15. *Let $f : X \to Y$ be a function. Then f is* **surjective** *(or a surjection) if the range of f coincides with its codomain, i.e.*

$$\text{ran} f = Y.$$

Note: surjection is also called "onto".



`Proving surjection:`

We know that for all $f : X \to Y$: ____________________________

Thus, *to show that $f : X \to Y$ is a surjection it is sufficient to prove that* ____________________

In other words,
to prove that $f : X \to Y$ is a surjective function it is sufficient to show that ______________________________
____

Question: How to disprove surjectivity?

EXAMPLE 16. *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to [0, \infty)$ defined by $f(x) = g(x) = x^4$. Determine whether the following are true*

(a) $\text{ran} f = \text{ran} g$

(b) $f = g$

(c) *f is surjective*

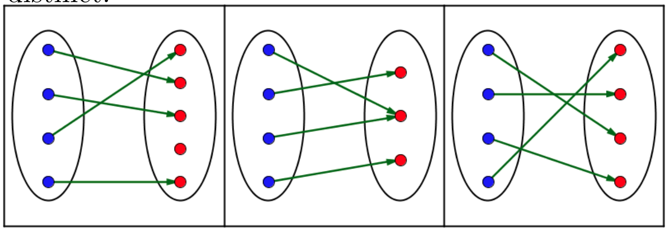
(d) *g is surjective*

EXAMPLE 17. *Prove that the function $f : \mathbb{R} - \{2\} \to \mathbb{R} - \{3\}$ defined by $f(x) = \dfrac{3x}{x-2}$ is surjective.*

Injective functions ("one to one")

DEFINITION 18. *Let $f : X \to Y$ be a function. Then f is* **injective** *(or an injection) if whenever $x_1, x_2 \in X$ and $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$.*

In other words, f is injective if and only if the ranges of every two distinct points in the domain of f are distinct.



EXAMPLE 19. *Given $X = \{1, 2, 3\}$ and $Y = \{3, 4, 5\}$. Determine whether the following functions are injective. Justify your answer.*

(a) $f : X \to Y$ *defined by* $G_f = \{(1, 3), (2, 4), (3, 5)\}$

(b) $g : X \to Y$ *defined by* $G_g = \{(1, 5), (2, 4), (3, 4)\}$

`Proving injection:`
Let $P(x_1, x_2) : x_1 \neq x_2$ *and* $Q(x_1, x_2) : f(x_1) \neq f(x_2)$.

Then by definition f is injective if ____________________.

Using contrapositive, we have ______________________________________.

In other words, to prove injection show that:

Question: How to disprove injectivity?

EXAMPLE 20. *Prove or disprove injectivity of the following functions.*

(a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sqrt[5]{x}$.

(b) $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^4$.

(c) $f : \mathbb{Z} \to \mathbb{Z},\ f(n) = \begin{cases} n/2 & \text{if} \quad n \in \mathbb{E}, \\ 2n & \text{if} \quad n \in \mathbb{O}. \end{cases}$

(d) $f : \mathbb{Z} \to \mathbb{Z},\ f(n) = \begin{cases} n & \text{if} \quad n \in \mathbb{E}, \\ 5n & \text{if} \quad n \in \mathbb{O}. \end{cases}$

`Discussion Exercise.`

- Must a strictly increasing or decreasing function be injective?

- Must an injective function be strictly increasing or decreasing?

EXAMPLE 21. *Prove or disprove injectivity of the following functions. In each case, $f : \mathbb{R} \to \mathbb{R}$.*

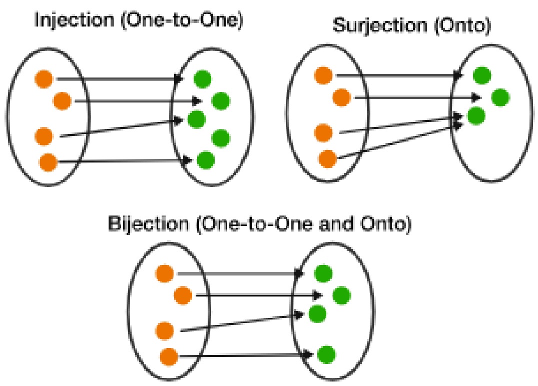
(a) $f(x) = 3x^5 + 5x^3 + 2x + \pi$.

(b) $f(x) = x^{12} + x^8 - x^4 + 12$.

Bijective functions

DEFINITION 22. *A function that is both surjective and injective is called* **bijective** *(or bijection.)*

f is not bijective $\Leftrightarrow$ ____



PROPOSITION 23. *A function f is bijective if and only if every point in* codom$f$ *has a* <u>*unique*</u> *preimage in the* dom$f$.

EXAMPLE 24. *Determine which of the following functions are bijective.*

(a) $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^3$.

(b) $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$.

PROPOSITION 25. *Let $f : A \to B$ and $g : B \to C$. Then*

i. *If f and g are surjections, then $g \circ f$ is also a surjection.*

`Proof.`

ii. *If f and g are injections, then $g \circ f$ is also an injection.*

`Proof.`

COROLLARY 26. *If f and g are bijections, then $g \circ f$ is also a bijection.*

Identity Function

For a set X we define the *identity function* $I_X : X \to X$ by the rule $I_X(x) = x$ for all $x \in X$. In other words, the identity function maps every element to itself.

Though this seems like a rather trivial concept, it is useful and important.

PROPOSITION 27. *Let* $f : X \to Y$. *Then* $f \circ I_X = f$ *and* $I_Y \circ f = f$.

5.4 Invertible Functions

Inverse Functions

DEFINITION 28. *Let* $f : X \to Y$ *be a function. We say that* f *is* **invertible** *if there is a function* $g : Y \to X$ *such that for all* $x \in X$ *and for all* $y \in Y$,

$$y = f(x) \quad \Leftrightarrow \quad x = g(y).$$

We say that such a function g *is an* ***inverse function*** *of* f.

Question 1 What is the inverse of g?

Question 2 Are the functions in Example 6 invertible?

REMARK 29. f is invertible if and only if its inverse is invertible.

EXAMPLE 30. *Show that the function* $f : \mathbb{R} - \{2\} \to \mathbb{R} - \{3\}$ *defined by* $f(x) = \dfrac{3x}{x-2}$ *is invertible and find its inverse function. (Note that the given function is bijective.)*

PROPOSITION 31. *A function $f : X \to Y$ is invertible if and only if there exists a function $g : Y \to X$ such that*

$$g \circ f = I_X \quad \text{and} \quad f \circ g = I_Y.$$

PROPOSITION 32. *The inverse function is unique.*

`Proof.`

Notation

When $f : X \to Y$ is invertible, the unique inverse function is denoted by f^{-1}, and $f^{-1} : Y \to X$.

REMARK 33. Finding the inverse of a bijective function is not always possible by algebraic manipulations. For example,

if $f(x) = e^x$ then $f^{-1}(x) =$ ________

The function $f(x) = 3x^5 + 5x^3 + 2x + 220$ is known to be bijective, but there is no way to find expression for its inverse.

THEOREM 34. *A function $f : X \to Y$ is invertible if and only if f is bijective.*

COROLLARY 35. *If a function $f : X \to Y$ is bijective, so is f^{-1}.*