

5.5 FUNCTIONS AND SETS

Image of a Set

DEFINITION 1. Let $f : X \rightarrow Y$ be a function. If $A \subseteq X$, we define $f(A)$, the **image** of A under f , by

$$f(A) = \{y \in Y \mid y = f(x) \text{ for some } x \in A\}.$$

EXAMPLE 2. Let $f : X \rightarrow Y$. Complete:

(a) If $A \subseteq X$ then $f(A) \subseteq \underline{\hspace{2cm}} \subseteq \underline{\hspace{2cm}}$

(b) $f(X) = \underline{\hspace{2cm}}$

(c) $y \in f(A) \Leftrightarrow \underline{\hspace{4cm}}$

(d) $y \notin f(A) \Leftrightarrow \underline{\hspace{4cm}}$

EXAMPLE 3. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = \cos x$. Find the following

(a) $f([-\pi/2, \pi/2]) =$

(b) $f([-\pi/2, 0]) =$

(c) $f([-\pi/2, \pi/2017]) =$

(d) $f([-\pi/4, \pi/4]) =$

(e) Does $-\frac{1}{2017}$ belong to $f([-\pi/2, \pi/2])$?

(f) $f(0) =$, $f(\{0\}) =$, $f(\{-\pi/4, \pi/4\}) =$

Inverse Image

DEFINITION 4. Let $f : X \rightarrow Y$ be a function and let B be a subset of its codomain (i.e. $B \subseteq Y$). Then the **inverse image** of B (written $f^{-1}(B)$) is the set

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

The inverse image $f^{-1}(B)$ is a subset of domain of f containing all preimages of points from B under f .

EXAMPLE 5. Let $f : X \rightarrow Y$ and $B \subseteq Y$. Complete:

(a) $f^{-1}(B) \subseteq$ _____

(b) $x \in f^{-1}(B) \Leftrightarrow$ _____

(c) $x \notin f^{-1}(B) \Leftrightarrow$ _____

(d) If $x \in X$ and $y \in Y$, then

$$f(x) \text{ _____ } Y, \quad f(\{x\}) \text{ _____ } Y, \quad f^{-1}(y) \text{ _____ } X, \quad f^{-1}(\{y\}) \text{ _____ } X,$$

EXAMPLE 6. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x^4$. Find the following:

(a) $f([1, 2]) =$

(b) $f([-2, -1]) =$

(c) $f^{-1}([1, 16]) =$

(d) $f^{-1}([-16, -1]) =$

(e) $f^{-1}([-1, 1]) =$

(f) $f^{-1}(\mathbb{R}) =$

(g) $f^{-1}(\mathbb{R}^+) =$

(i) $f^{-1}(\{16\}) =$ _____ , $f^{-1}(\{-16\}) =$ _____ , $f^{-1}(\{-16, 16\}) =$ _____ , $f^{-1}(16) =$ _____

EXAMPLE 7. Let $A \subseteq \text{dom}(f)$ and $B \subset \text{codom}(f)$. Determine the truth or falsehood of the following

(a) $f^{-1}(f(A)) = A$

(b) $f(f^{-1}(B)) = B$

EXAMPLE 8. Let $A_1, A_2 \subseteq \text{dom}(f)$. Determine the truth or falsehood of the following

$$f(A_1) \cap f(A_2) = f(A_1 \cap A_2)$$

Summary

Let $f : X \rightarrow Y$ and $A \subseteq X$ and $B \subseteq Y$. Then above definitions imply the following tautologies

- $\forall y \in Y, ((y \in \text{ran}(f)) \Leftrightarrow (\exists x \in X \ni f(x) = y))$.
- $\forall y \in Y, ((y \in f(A)) \Leftrightarrow (\exists x \in A \ni f(x) = y))$.
- $\forall x \in X, ((x \in f^{-1}(B)) \Leftrightarrow (f(x) \in B))$.

Also note that

- If $B \subseteq \text{ran}(f)$ then $(S = f^{-1}(B)) \Rightarrow (f(S) = B)$.

EXAMPLE 9. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = 3x + 4$ and let $B = \{x \in \mathbb{R} | x > 0\}$. Find $f^{-1}(B)$. Justify your answer (give a formal proof).

EXAMPLE 10. $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f(x) = 5x - 4$. Find $f([0, 1])$. Justify your answer (give a formal proof).

PROPOSITION 11. Let $f : X \rightarrow Y$. If $A_1 \subseteq A_2 \subseteq X$ then $f(A_1) \subseteq f(A_2)$.

Proof.

PROPOSITION 12. Let $f : X \rightarrow Y$. If B_1 and B_2 are subsets of Y then

$$f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2).$$

Proof.

EXAMPLE 13. (cf. Example 8.) Let $f : X \rightarrow Y$ and A_1 and A_2 be subsets of X .

(a) Prove that $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$.

(b) Prove that if, in addition, f is an injective function, then $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$.

EXAMPLE 14. (cf. Example 7.) Let $f : X \rightarrow Y$ and B be a subset of Y .

(a) Prove that $f(f^{-1}(B)) \subseteq B$.

(b) Prove that if, in addition, f is an surjective function, then $f(f^{-1}(B)) = B$.