

## 14: Nonhomogeneous Equations. Method of Undetermined Coefficients (section 3.5)

1. If  $y_1(t)$  and  $y_2(t)$  are two solutions of a second-order *nonhomogeneous* linear ODE,

$$y'' + p(t)y' + q(t)y = g(t), \quad g(t) \neq 0.$$

Then  $y_1(t) - y_2(t)$  is a particular solution of the corresponding homogeneous equation,

$$y'' + p(t)y' + q(t)y = 0.$$

2. *THEOREM: Solution of Nonhomogeneous Linear Equation*

Let

$$y'' + p(t)y' + q(t)y = g(t), \quad g(t) \neq 0$$

be a second-order *nonhomogeneous* linear differential equation. If  $y_p(t)$  is a particular solution of this equation and  $y_h(t)$  is the general solution of the corresponding homogeneous equation,

$$y'' + p(t)y' + q(t)y = 0,$$

then

$$y(t) = y_h(t) + y_p(t)$$

is the general solution of the nonhomogeneous equation.

3. To solve a nonhomogeneous linear ODE:

**Step 1:** Find a particular solution of a nonhomogeneous linear ODE.

**Step 2:** Find general solution of the corresponding homogeneous linear ODE.

**Step 3:** Add the results of Steps 1&2.

4. One solution of  $y'' - y = t$  is  $y(t) = -t$ , as you can verify. What is the general solution?

5. Consider

$$y'' + p(t)y' + q(t)y = g_1(t) + g_2(t) \quad (1)$$

. If  $y(t) = y_p(t)$  and  $y(t) = Y_p(t)$  are particular solutions of

$$y'' + p(t)y' + q(t)y = g_1(t)$$

and

$$y'' + p(t)y' + q(t)y = g_2(t),$$

respectively, then  $y(t) = y_p(t) + Y_p(t)$  is a particular solution of (1).

### Method of Undetermined Coefficients

6. Consider a particular class of nonhomogeneous linear ODE with constant coefficients

$$ay'' + by' + cy = g(t),$$

where  $a, b, c$  are real constants and  $g(t)$  involves linear combinations, sums and products of

$$t^m, \quad e^{\alpha t}, \quad \sin(\beta t), \quad \cos(\beta t).$$

7. The idea of the Method of Undetermined Coefficients: guess a particular solution  $y_p(t)$  using a generalized form of  $g(t)$ . Then by substitution determine the coefficients for the generalized solution.

8. Find general solution:

(a)  $y'' - 3y' + 2y = 4e^{3t}$

$$(b) \quad y'' - 3y' + 2y = 4e^t$$

$$(c) \quad y'' + 10y' + 25y = 3e^{-5t}$$

### 9. Multiplicity $s$ of a given number $\alpha + i\beta$

If  $g(t) = (B_0t^n + B_1t^{n-1} + \dots + B_n)e^{\alpha t}$ ,  $\beta = 0$ , then

- $\alpha$  doesn't coincide with a root of characteristic polynomial  $\Rightarrow s = 0$ .
- $\alpha$  coincides with a non repeated root of characteristic polynomial  $\Rightarrow s = 1$ .
- $\alpha$  coincides with a repeated root of characteristic polynomial  $\Rightarrow s = 2$ .

If  $g(t) = (B_0t^n + B_1t^{n-1} + \dots + B_1t^{n-1} + \dots + B_n)e^{\alpha t} \cos(\beta t)$  or  $g(t) = (B_0t^n + B_1t^{n-1} + \dots + B_n)e^{\alpha t} \sin(\beta t)$ ,  $\beta \neq 0$ , then

- $\alpha + i\beta$  doesn't coincide with a root of characteristic polynomial  $\Rightarrow s = 0$ .
- $\alpha + i\beta$  coincides with a (complex) root of characteristic polynomial  $\Rightarrow s = 1$ .

10. If  $g(t) = (B_0t^n + B_1t^{n-1} + \dots + B_n)e^{\alpha t} \cos(\beta t)$  or  $g(t) = (B_0t^n + B_1t^{n-1} + \dots + B_n)e^{\alpha t} \sin(\beta t)$  then we choose a particular solution in the form

$$y_p(t) = t^s(A_0t^n + A_1t^{n-1} + \dots + A_n)e^{\alpha t} \cos(\beta t) + t^s(D_0t^n + D_1t^{n-1} + \dots + D_n)e^{\alpha t} \sin(\beta t),$$

respectively.

11. Find general solution of

$$y'' + 2y' + 5y = 3 \sin 2t$$

12. Determine a suitable choice for  $y_p$ :

	$ay'' + by' + cy = g(t)$	$r_1, r_2$	$\alpha + i\beta$	$s$	$y_p(t)$
1	$y'' = e^t$				
2	$y'' - 4y' + 4 = e^{2t}$				
3	$y'' + 2y' + 10y = 6 \sin(3t)$	$r_{1,2} = -1 \pm 3i$			
4	$y'' + 2y' + 10y = 6e^{-t} \sin(3t)$	$r_{1,2} = -1 \pm 3i$			
5	$y'' + 2y' + 10y = (t^3 - 1) \sin(3t)$	$r_{1,2} = -1 \pm 3i$			
6	$y'' + 2y' + 10y = t^2 e^{-t} \sin(3t)$	$r_{1,2} = -1 \pm 3i$			
7	$y'' = 2t - 2013$	$r_1 = r_2 = 0$			