27: Complex Eigenvalues (section 7.6)

1. When the characteristic equation has real coefficients, complex eigenvalues always appear in conjugate pairs.

2. Case $n = 2$. If $\lambda = \alpha + i\beta$ is a complex eigenvalue of the coefficient matrix A in the homogeneous system $X' = AX$ and v is a corresponding eigenvector then

   - $$\left\{e^{\lambda t}v, e^{\bar{\lambda}t}\bar{v}\right\}$$
     is a fundamental set of solutions of the system $X' = AX$.

   - $$\left\{\mathrm{Re}(e^{\lambda t}v), \mathrm{Im}(e^{\lambda t}v)\right\}$$
     is a **real** fundamental set of solutions of the system $X' = AX$.

3. Example. *Consider* $\begin{pmatrix} 3 & 1 \\ -5 & 1 \end{pmatrix}$

   (a) *Find general solution of the system* $X' = AX$.

   (b) *Find general real solution of the system* $X' = AX$.

   (c) *Find solution subject to the initial conditions* $x_1(0) = 2, x_2(0) = 3$.

4. If $\lambda = \alpha + i\beta$ is a complex eigenvalue of a real matrix A and $v = a + ib$ is a corresponding eigenvector, then

$$\mathrm{Re}(e^{\lambda t}v) = e^{\alpha t}(a\cos(\beta t) - b\sin(\beta t)), \quad \mathrm{Im}(e^{\lambda t}v) = e^{\alpha t}(a\sin(\beta t) + b\cos(\beta t)).$$

5. Case $n = 3$. If $\alpha \pm i\beta$ are complex eigenvalue of the coefficient matrix A, then the third eigenvalue must be real (denote it by λ). Let v and w be eigenvectors corresponding to $\alpha + i\beta$ and λ, respectively. Then

$$\left\{\mathrm{Re}(e^{\lambda t}v), \mathrm{Im}(e^{\lambda t}v), e^{\lambda t}w\right\}$$

is a **real** fundamental set of solutions of the system $X' = AX$.