29: The Phase Plane: Linear Systems (section 9.1)

1. We consider here a non-singular 2×2 matrix A ($\det A \neq 0$). In this case $AX = 0$ implies $X = 0$. Points where $AX =$ correspond to the equilibrium (constant) solutions of system $X' = AX$, and they are called *critical points*. It follows that $X = 0$ is the only critical point of the system system $X' = AX$.

2. Solution of system $X' = AX$ are combinations of eigenvectors v_1, v_2 with coefficients depending on the parameter t. This solution is also a vector functions of t. Such functions can be viewed as a parametric representation for a curve in the x_1x_2-plane. We regard to this curve as the path, or **trajectory**, traversed by a moving particle whose velocity $X'(t)$ is specified by the differential equation. The plane x_1x_2 itself is called the **phase plane**, and a representative set of trajectories is referred to as a **phase portrait**.

Case 1. Real Distinct Eigenvalues

3. General solution (λ_1, λ_2 are eigenvalues and v_1, v_2 are corresponding eigenvectors):

$$X(t) = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2. \tag{1}$$

Coordinates of $X(t)$ in the basis $\{v_1, v_2\}$ are

$$(C_1 e^{\lambda_1 t}, C_2 e^{\lambda_2 t}) =: (\xi_1(t), \xi_2(t)).$$

Eliminating the parameter t, one get

$$\xi_2 = C \xi_1^{\lambda_2/\lambda_1.}$$

Case 1a: Real Distinct Eigenvalues of the Same Sign

4. Example. Sketch several trajectories in the phase plane for the system

$$\begin{aligned} x_1' &= -2x_1 + x_2 \\ x_2' &= 2x_1 - 3x_2 \end{aligned} \tag{2}$$

Previously we obtained[1]

$$\lambda_1 = -4, \quad \lambda_2 = -1, \quad v_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

5. Sketch several trajectories in the phase plane when both eigenvalues are positive.

[1] see set 26(#3) of notes

Case 1b: Real Eigenvalues of the Opposite Sign

6. Example. Sketch several trajectories in the phase plane for the system

$$\begin{cases} x_1' &= x_1 + 2x_2 \\ x_2' &= 4x_1 + 3x_2 \end{cases} \tag{3}$$

Previously we obtained[2]

$$\lambda_1 = -1, \quad \lambda_2 = 5, \quad v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Case 2: Complex Eigenvalues

7. General solution ($\lambda = \alpha + i\beta$ is a complex eigenvalue and $v = a + ib$ is a corresponding eigenvector):

$$X(t) = C_1 \mathrm{Re}(e^{\lambda t} v) + C_2 \mathrm{Im}(e^{\lambda t} v) = C_1 e^{\alpha t}(a\cos(\beta t) - b\sin(\beta t)) + C_2 e^{\alpha t}(a\sin(\beta t) + b\cos(\beta t)).$$

8. Sketch several trajectories in the phase plane in the case $\alpha = 0$ (i.e. λ is pure imaginary).
9. Sketch several trajectories in the phase plane in the case $\alpha < 0$.
10. Sketch several trajectories in the phase plane in the case $\alpha > 0$

Case 3: Repeated Eigenvalues

Case 3a: There is Basis of Eigenvectors

11. General solution (λ is eigenvalue and v_1, v_2 are corresponding eigenvectors):

$$X(t) = C_1 e^{\lambda t} v_1 + C_2 e^{\lambda t} v_2.$$

Coordinates of $X(t)$ in the basis $\{v_1, v_2\}$ are

$$(C_1 e^{\lambda t}, C_2 e^{\lambda t}) =: (\xi_1(t), \xi_2(t)).$$

Eliminating the parameter t, one get

$$\xi_2 = \frac{C_2}{C_1} \xi_1.$$

12. Sketch several trajectories in the phase plane in this case.

[2] see Homework 13 (#1, Spring 2013)

Case 3b: There is NO Basis of Eigenvectors

13. General solution (λ is eigenvalue of multiplicity 2, v is a corresponding eigenvector, and w is a generalized eigenvector):

$$X(t) = C_1 e^{\lambda t} v + C_2 (t e^{\lambda t} v + e^{\lambda t} w)$$

14. Example. Sketch several trajectories in the phase plane for the system

$$\begin{aligned} x_1' &= -3x_1 + \tfrac{5}{2}x_2 \\ x_2' &= -\tfrac{5}{2}x_1 + 2x_2 \end{aligned}$$

Previously we obtained[3]

$$\lambda = -\frac{1}{2}, \quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad w = \begin{pmatrix} 0 \\ 2/5 \end{pmatrix}.$$

Summary

15. Stability properties of linear systems $X' = AX$ with $\det(A - \lambda I) = 0$ and $\det A \neq 0$.

Eigenvalues, λ	Type of Critical Point	Stability
$\lambda_1 > \lambda_2 > 0$	Proper node	Unstable
$\lambda_1 < \lambda_2 < 0$	Proper node	Asymptotically stable
$\lambda_2 < 0 < \lambda_1$	Saddle point	Unstable
$\lambda_{1,2} = \alpha \pm i\beta$, $\alpha > 0$	Spiral source	Unstable
$\lambda_{1,2} = \alpha \pm i\beta$, $\alpha < 0$	Spiral sink	Asymptotically stable
$\lambda_{1,2} = \alpha \pm i\beta$, $\alpha = 0$	Center	Stable
$\lambda_1 = \lambda_2 > 0$	Proper or Improper node	Unstable
$\lambda_1 = \lambda_2 < 0$	Proper or Improper node	Asymptotically stable

[3] see set 28(#10) of notes