8. Exact Equations and Integrating Factors (section 2.6)

1. Method for solving the first order ODE
$$M(x,y)+N(x,y)y'=0 \quad (\text{or} \quad M(x,y)\mathrm{d}x+N(x,y)\mathrm{d}y=0) \tag{1}$$
for the special case in which (1) represents the **exact differential** of a function $z=\Phi(x,y)$.

2. The equation (1) is an **exact ODE** if there exists a function $\Phi(x,y)$ having continuous partial derivatives such that
$$\Phi_x(x,y)=M(x,y), \qquad \Phi_y(x,y)=N(x,y).$$
and The general solution of the equation is $\Phi(x,y)=C$ (geometrically, the integral curve $y=y(x)$ lies on a level curve of the function $z=\Phi(x,y)$.)

3. *TEST for Exactness:* If M and N have continuous partial derivatives then the ODE (1) is exact if and only if
$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \tag{2}$$

4. Question: *Is every separable equation exact?*

5. *Determine whether the following ODE are exact:*
   (a) $3x^2-2xy+2+(6y^2-x^2+3)y'=0$
   (b) $(3x^2y+2xy+y^3)\mathrm{d}x+(x^2+y^2)\mathrm{d}y=0$

6. `Test for exactness and Conservative Vector Field`:
The test for exactness is the same as the test for determining whether a vector field $\mathbf{F}(x,y)=\langle M(x,y),N(x,y)\rangle$ is conservative. Namely, it is the same as the test for determining whether $\mathbf{F}(x,y)=$ the gradient of a potential function.

   `CONCLUSION`: *A general solution to an exact differential equation can be found by the method used to find a potential function for a conservative vector field.*

7. Solve $3x^2-2xy+2+(6y^2-x^2+3)y'=0$

8. **A nonexact ODE made exact:** If ODE is not exact, it may be possible to make it exact by multiplying by an appropriate integrating factor.
   - If $\dfrac{M_y-N_x}{N}$ is a function of x alone, then an integrating factor for (1) is
   $$\mu(x)=e^{\int\frac{M_y-N_x}{N}\mathrm{d}x}.$$
   - If $\dfrac{N_x-M_y}{M}$ is a function of y alone, then an integrating factor for (1) is
   $$\mu(y)=e^{\int\frac{N_x-M_y}{M}\mathrm{d}y}.$$

9. Solve $(3x^2y+2xy+y^3)\mathrm{d}x+(x^2+y^2)\mathrm{d}y=0$