# Continuity and Intermediate Value Theorem

Prepare to embark on a magical journey of calculus.
Choose your weapon of choice...

Jayci Blake, Tommy Dunham, and Jennifer Sakowski

# Introduction

* Continuity is one of the most basic principles of calculus
  * Continuity is required for a function to be differentiated or integrated.
  * Both the Fundamental Theorem of Calculus and the Mean Value Theorem rely on the concept of continuity

* Intermediate Value Theorem
  * If a function is continuous on the interval [a,b], it must pass through all points that exist between f(a) and f(b).

# History

"Calculus required continuity, and continuity was supposed to require the infinitely little; but nobody could discover what the infinitely little might be." -- Bertrand Russell

* Calculus
  * Newton and Leibnitz realized the need to explain continuity in order to fully explain limits

* Continuity
  * Euler provided the earliest definition
  * Cauchy provided the first modern definition of continuity in the early $19^{th}$ century (published in 1821 book)

Augustin-Louis **Cauchy**

# Definition of Continuity

* "The function f(x) will be, between two assigned values of the variable x, a continuous function of this variable if for each value of x between these limits, the absolute value of the difference f(x+a)-f(x) decreases indefinitely with a." *Cours d'analyse* (Oeuvres II.3, p.43) -**Augustin-Louis Cauchy**

* If you don't have to pick up your pencil when drawing the function, it is continuous

* f(x) is continuous if f(a) exists and

  $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = \lim_{x \to a} f(x) = f(a)$

  * If the functions f and g are continuous at c, then the following are also continuous at c:
    * f + g
    * f – g
    * fg
    * f/g if $g(c) \neq 0$

# Applications of Continuity

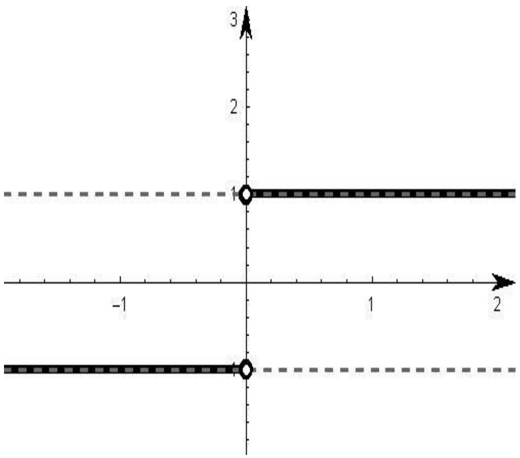
- Discontinuities can represent significant events in the data
  - For example, when points are given to the houses of Hogwarts, the tallying hourglasses fill with rubies, emeralds, sapphires, and yellow gemstones in discrete amounts, not continuously.
  - When Hermione answered Professor McGonagall's question and received 10 points for Gryffindor, this was a jump discontinuity.

# Applications of Continuity

* Velocity as a function of time, weight, and pressure versus temperature graphs are all examples of continuous functions
  * For example, the path of Iron Man's flight trajectory as he defended New York City from the wrath of Loki was a continuous function of height versus time

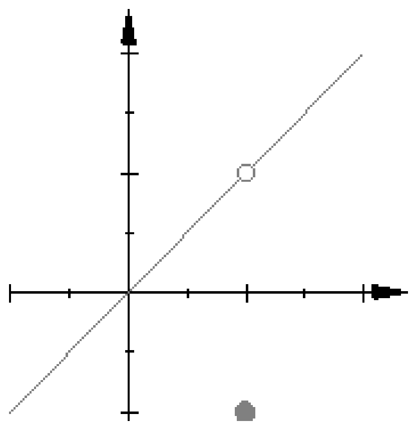
# Discontinuities

## Jump



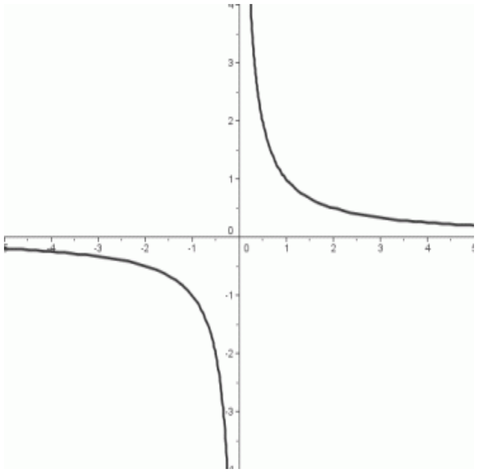

$$f(x)=\begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

## Removable



$$f(x)=\begin{cases} x, & x \neq 2 \\ -2, & x = 2 \end{cases}$$

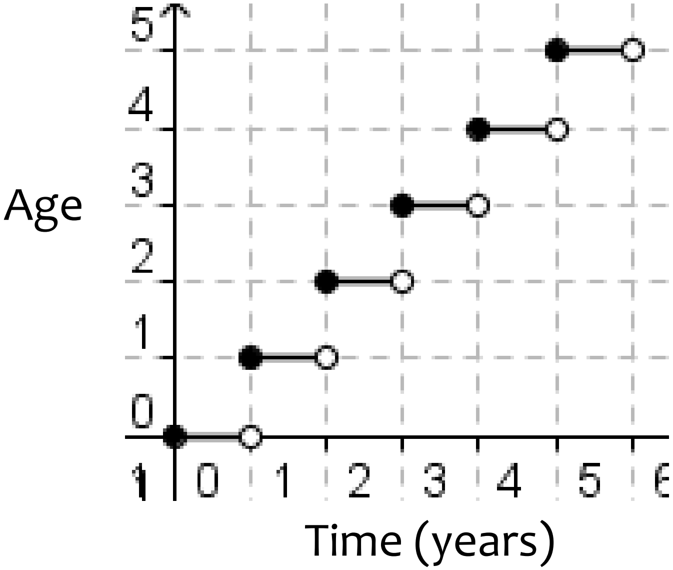
## Infinite



$$f(x) = \frac{1}{x}$$

# Example 1

* f(x) models Bilbo Baggins's age as a function of time
  * i.e. He was not officially 111 until exactly 12 months after his 110th birthday, even though his biological age was continuously increasing during those 12 months.



# Example 2

- When the members of the Fellowship of the Ring are escaping the Mines of Moria, the pathway crumbles, forcing Frodo and Aragorn to jump across the hole to safety.
- Given the function $f(x) = \frac{x^4 - x^3 + 4x - 4}{x - 1}$ to represent this flee, what type of discontinuity would be expected and where would it be found?
  - $f(x) = \frac{x^4 - x^3 + 4x - 4}{x - 1}$
  - $f(x) = \frac{(x^3 + 4)(x - 1)}{x - 1}$

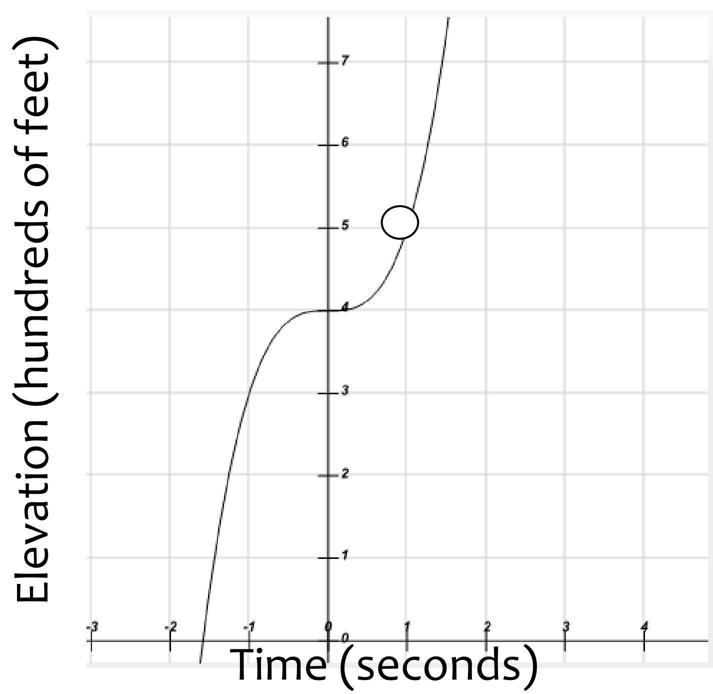
    with a hole at x=1

    (Use synthetic or long division)
  - Lim f(x) = lim f(x)

    x->1+ x->1-
  - lim f(x) exists, but f(1) DNE,

    x->1

    so a removable discontinuity occurs at x=1

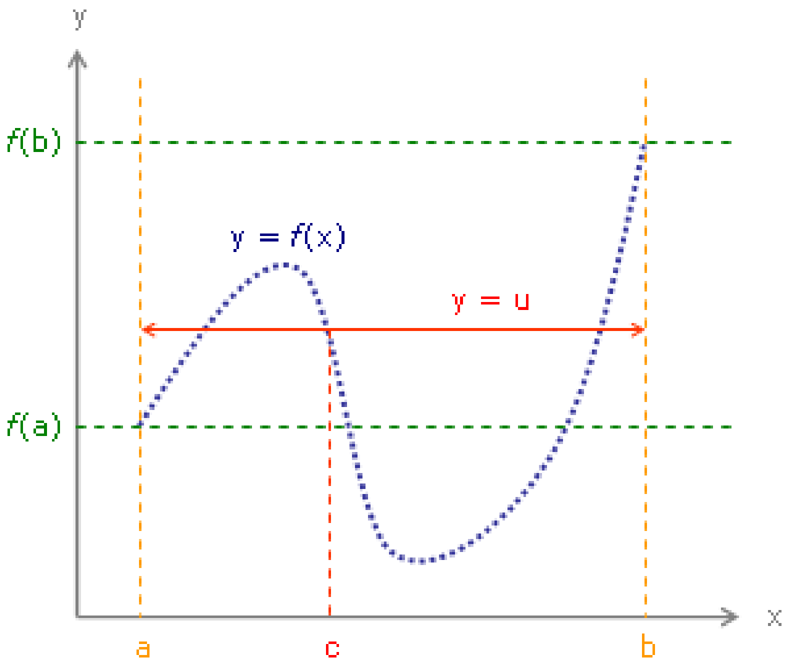


# History

* Intermediate Value Theorem
  * Bolzano was a Roman Catholic priest that was dismissed for his unorthodox religious views.
  * This led to him developing theories of philosophy and mathematics for the remainder of his life.
  * Bernard Bolzano provided a proof in his 1817 paper.
  * His theorem was created to formalize the analysis of functions and expound upon the work of Lagrange

# Intermediate Value Theorem

* If f(x) is continuous on the closed interval [a,b] and N is any number strictly between f(a) and f(b), then there is a number c, a<c<b, so that f(c)=N.




This theory will be famous. There won't be a calculus student in our world who doesn't know its name.

# Example 3

* Sam and Frodo's journey to destroy the One Ring follows the elevation path (in thousands of feet) given by the function $f(t) = -t^3 + 2t^2 + 3t$. Their two day journey from the gates of Moria (at sea level) to the other side of the Misty Mountains 6,000 ft higher. Did they pass over the Bridge of Khazad-dûm as they made their way through the Mines of Moria, which was at an elevation of 4,000 ft?

# Example 3 Work

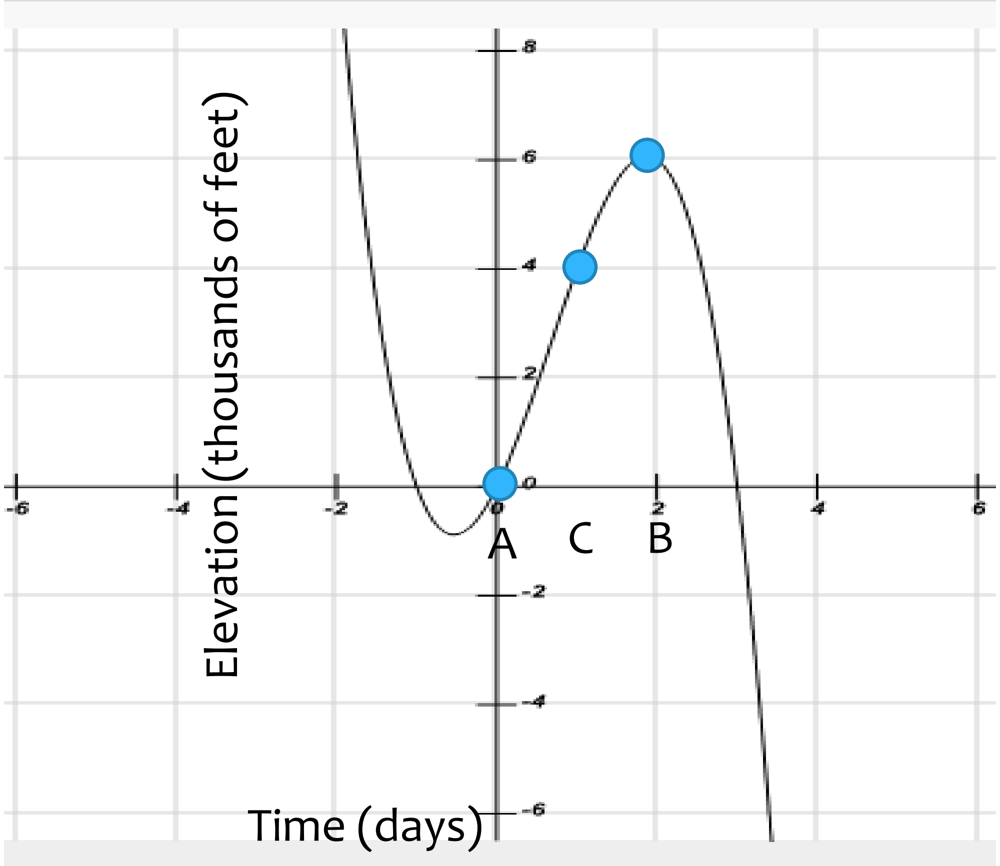
* Prove there is a number c on [0,2] that shows they passed over the Bridge of Khazad-dûm at f(c)=4.
* The function f(t) is continuous on [0,2]
* Find f(0) and f(2)
  * $f(0) = -0^3 + 2(0)^2 + 3(0)$=0
  * $f(2) = -(2)^3 + 2(2)^2 + 3(2)$=6
* Since 0<4<6 there is a number c on [0,2] such that f(c)=4 by IVT
* Therefore, Sam and Frodo did pass over the Bridge of Khazad-dûm on their way through the Mines of Moria.

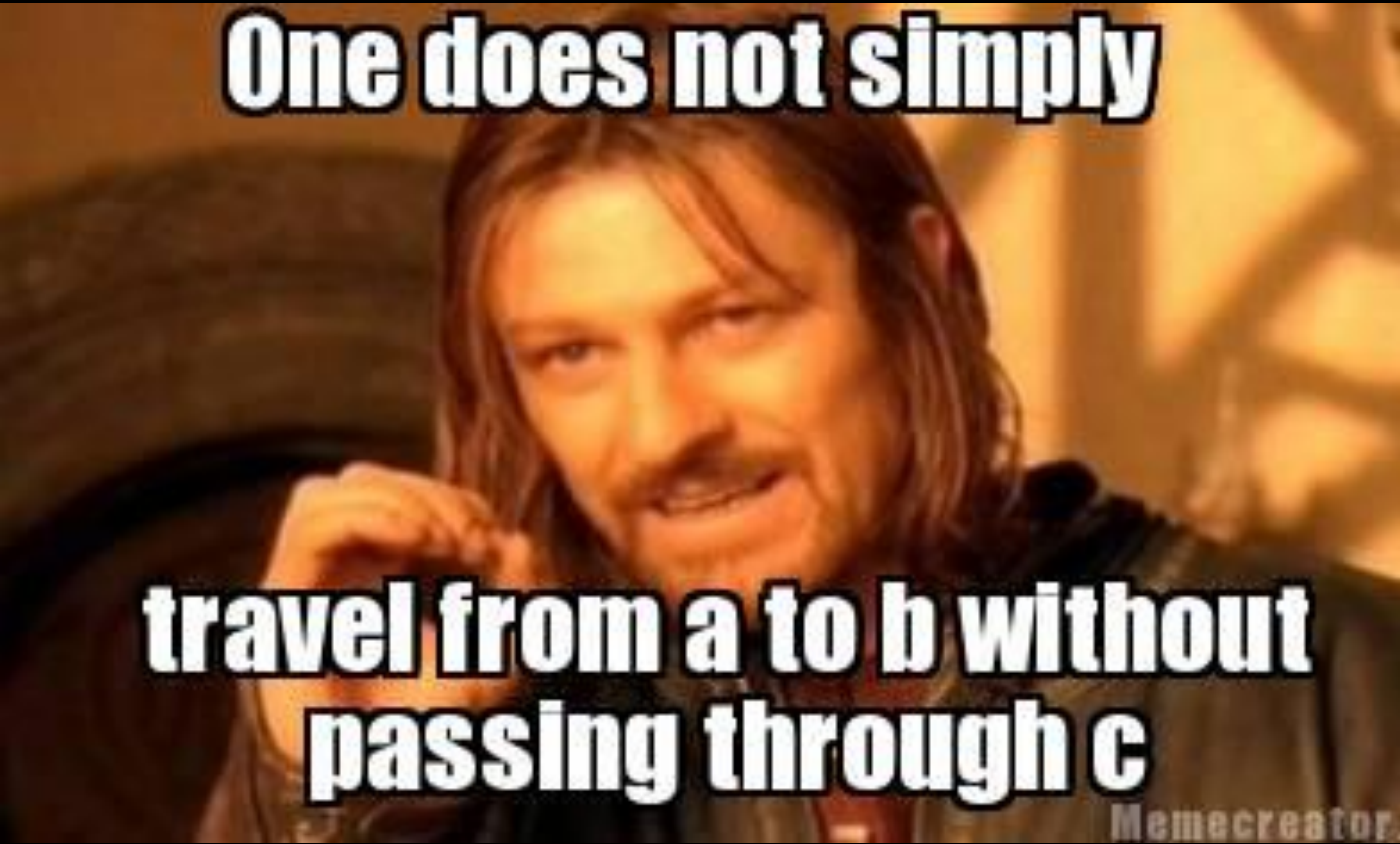
# Example 3 (cont.)

A-West Gate

B-Other side of Misty Mountains

C- Bridge of Khazad-dûm




One does not simply
travel from a to b without
passing through c
Memecreator.

"Dark problems lie ahead of us and there will be a time when we must choose between what is easy and what is right."

# References


- Grabiner, Judith V. "Who Gave You the Epsilon? Cauchy and the Origins of Rigorous Calculus." *The American Mathematical Monthly* 90.3 (1983): 185-94.*Mathematical Association of America*. Web. 20 Nov. 2012.<http://www.maa.org/pubs/Calc_articles/ma002.pdf>.
- Stewart, James. *Calculus: Early Vectors*. 1st ed. Pacific Grove: Brooks/Cole, 1999. Print.
- Weisstein, Eric W. "Bolzano's Theorem." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/BolzanosTheorem.html