MATH 308 Homework assignment 2 (Solutions)

Problem 1. First find the general solution

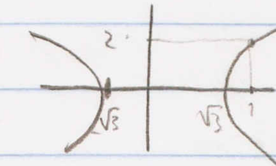
$$\underbrace{y' = \frac{x}{y}}_{\text{separable}} \iff y\,dy = x\,dx \iff \int y\,dy = \int x\,dx \Rightarrow$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C_1 \iff$$

$$y^2 - x^2 = C \rightarrow \text{hyperbolas}$$

(a) If $y(2) = 1 \Rightarrow \overbrace{1}^{y_0^2} - \overbrace{4}^{x_0^2} = -3 \Rightarrow C = -3 \Rightarrow$

$$y^2 - x^2 = -3 \iff x^2 - y^2 = 3$$



$$y = \pm\sqrt{x^2-3}$$

but since $y(2) = 1 > 0$ then

$$\boxed{y = \sqrt{x^2-3}}$$

The interval, in which the solution is defined, is $\boxed{x > \sqrt{3}}$

(b) If $y(2) = -1$ then as in (a) $C = -3 \Rightarrow$

$$y = \pm\sqrt{x^2-3} \quad \text{but} \quad y(2) = -1 < 0 \Rightarrow$$

$$\boxed{y = -\sqrt{x^2-3}}$$

The interval, in which the solution is defined, is $\boxed{x > \sqrt{3}}$

(c) If $y(1) = 2 \Rightarrow \underbrace{4}_{y_0^2} - \underbrace{1}_{x_0^2} = 3 \Rightarrow C = 3 \Rightarrow$

$$y^2 - x^2 = 3 \Rightarrow y^2 = x^2 + 3 \Rightarrow y = \pm\sqrt{x^2+3}$$

but since $y(1) > 0$ then $y = \sqrt{x^2+3}$ and the interval in which the solution is defined, is the whole $\mathbb{R}$

Problem 2

$$ty' + 3y = \cos t \quad , \quad t > 0$$

Use the method of integrating factor

$$y' + \frac{3}{t}y = \frac{\cos t}{t}$$

$$\mu y' + \frac{3}{t}\mu y = \frac{\cos t}{t}\mu$$

Find μ s.t. $\mu' = \frac{3}{t}\mu \Rightarrow$ we can take $\mu = e^{\int \frac{3}{t}dt} = e^{3\ln t} = t^3$

$$\mu y' + \mu' y = (\underbrace{\mu}_{t^3} y)' = \frac{\cos t}{t} \cdot t^3 \Rightarrow$$

$$t^3 y = \int t^2 \cos t = t^2 \sin t + 2t\cos t - 2\sin t + C$$

$$\boxed{y(t) = \frac{\sin t}{t} + \frac{2\cos t}{t^2} - \frac{2\sin t}{t^3} + \frac{C}{t^3}}$$

$$\boxed{\lim_{t \to +\infty} y(t) = 0}$$

Calculation of the integral

$$\int t^2 \cos t \, dt \underset{\text{By part}}{=}$$

$$t^2 \sin t - 2\int t \sin t \, dt =$$

$$= t^2 \sin t - \left(-2t\cos t + 2\int \cos t \, dt\right) =$$

$$= t^2 \sin t + 2t\cos t - 2\sin t$$

Problem 3 (a) $\quad y' - 5y = te^{4t}$

Use the method of integrating factor

$$\mu y' \underbrace{- 5\mu}_{\mu'} y = te^{4t}\mu$$

$$\mu' = -5\mu \Rightarrow \mu = e^{-5t}$$

$$\underbrace{(\mu y)'}_{\mu y' + \mu' y} = te^{4t}e^{-5t} = te^{-t} \Rightarrow$$

$$\int te^{-t} = -te^{-t} + \int e^{-t} = -te^{-t} - e^{-t}$$

$$e^{-5t} y = -te^{-t} - e^{-t} + C \Rightarrow$$

$$y = -te^{4t} - e^{4t} + Ce^{5t}$$

$y(0)=a \Rightarrow a = -1 + C \Rightarrow C = a+1$

$$\boxed{y(t) = -te^{4t} - e^{4t} + (a+1)e^{5t}}$$

$$= -e^{4t}(t+1) + (a+1)e^{5t}$$

(b) Behavior at infinity:

$$y(t) = e^{4t}\left(-t - 1 + (a+1)e^{t}\right)$$

Everything depends on the sign of $a+1$

$$\boxed{\begin{array}{l} \text{If } a+1 > 0 \iff a > -1 \text{ then } y(t) \xrightarrow[t\to+\infty]{} +\infty \\ \text{If } a+1 \leq 0 \iff a \leq -1 \text{ then } y(t) \xrightarrow[t\to+\infty]{} -\infty \end{array}}$$

$a_0 = -1$

(c) If $a_0 = -1$ then $y(t) = -e^{4t}(t+1)$ and

$$\boxed{y(t) \xrightarrow[t\to+\infty]{} -\infty}$$

Problem 4 Let $Q(t)$ be the amount of salt in the tank at the time t.

$$Q'(t) = \text{rate in} - \text{rate out}$$

For $0 \leq t \leq 50$ min

$$\text{rate in} = \frac{1}{2} \cdot 4 \text{ lb/min} = 2 \text{ lb/min}$$

For $50 \leq t \leq 75$ min

$$\text{rate in} = 0$$

For any t rate out $= \dfrac{Q(t)}{100} \cdot 4$ lb/min $= \dfrac{Q}{25}$ lb/min

Therefore $Q(t)$ satisfies

(1) $Q(0)=0$

(2) $Q'(t) = 2 - \frac{Q}{25}$ for $0 \le t \le 50$

(3) $Q' = -\frac{Q}{25}$ for $50 \le t \le 75$

(2) $\Rightarrow$ $Q(t) = 50 + Ce^{-\frac{t}{25}}$

general solution

$Q(0)=0$ $\quad 0 = 50 + C \Rightarrow C = -50 \Rightarrow$ for $0 \le t \le 50$

$$Q(t) = 50\left(1 - e^{-\frac{t}{25}}\right) \Rightarrow$$

$$Q(50) = 50\left(1 - e^{-\frac{50}{25}}\right) = 50(1-e^{-2})$$

(3) $\Rightarrow$ $Q(t) = Ce^{-\frac{t}{25}}$ if $50 \le t \le 75$)

$$Q(50) = 50(1-e^{-2}) = Ce^{-2} \Rightarrow C = 50(1-e^{-2})e^{2} = 50(e^{2}-1)$$

$$Q(t) = 50(e^{2}-1)e^{-\frac{t}{25}} \quad 50 \le t \le 75 \Rightarrow$$

$$Q(75) = 50(e^{2}-1)e^{-3} = 50(e^{2}-1)e^{-3} = \boxed{50(e^{-1}-e^{-3})}$$

Problem 5 $\quad (9-t^2)y' + ty = 2t^2 \Leftrightarrow$

$$y' + \frac{t}{9-t^2}y = \frac{2t^2}{9-t^2}$$

$p(t) = \frac{t}{9-t^2}$, $g(t) = \frac{2t^2}{9-t^2}$

$p(t)$ and $g(t)$ are not continuous at $t=-3$ and $t=3$

a) If $t_0 = 1 \Rightarrow -3 < t_0 < 3 \overset{\text{Thm 2.4.1}}{\Rightarrow}$ the solution certainly exists on $\boxed{-3 < t < 3}$

b) If $t_0 = 4$ then $t_0 > 3 \Rightarrow$ the solution certainly exists for $\boxed{t > 3}$

c) If $t_0 = -5$ then $t_0 < -3 \Rightarrow$ the solution certainly exists for $\boxed{t < -3}$

Problem 6

(a) $y' = \frac{y}{x} + \frac{x}{y}$

Substitute $u = \frac{y}{x} \Rightarrow y = u \cdot x \Rightarrow$

$y' = u'x + u \Rightarrow$

$y' = \frac{y}{x} + \frac{x}{y} \Leftrightarrow u'x + \cancel{u} = \cancel{u} + \frac{1}{u} \Leftrightarrow u'x = \frac{1}{u} \Leftrightarrow$

$u\,du = \frac{dx}{x} \Leftrightarrow \frac{u^2}{2} = \ln|x| + C_1 \Rightarrow$

$u^2 = 2\ln|x| + C = \ln x^2 + C \Rightarrow$

$u = \pm\sqrt{\ln x^2 + C} \Rightarrow$

$y = xu = \boxed{\pm x\sqrt{\ln x^2 + C}}$

(b) $y' = \frac{x-y}{x+y} \Leftrightarrow y' = \frac{1-\frac{y}{x}}{1+\frac{y}{x}}$ → here we divide the numerator and denominator by x

Substitute $u = \frac{y}{x} \Rightarrow y = ux \Rightarrow$

$y' = u'x + u \Rightarrow u'x + u = \frac{1-u}{1+u} \Rightarrow xu' = \frac{1-u}{1+u} - u = \frac{1-2u-u^2}{1+u} \Rightarrow$

$-\frac{(1+u)\,du}{u^2+2u-1} = \frac{dx}{x} \Rightarrow -\int \frac{u+1}{u^2+2u-1}\,du = \ln|x| + C_1 \Rightarrow$

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$v = u^2+2u+1$

$dv = (2u+2)\,du$

$$-\frac{1}{2}\int\frac{dv}{v} = \ln|x| + C_1$$

$\Downarrow$

$$\ln|u^2+2u+1| + 2\ln|x| = C_2 \Rightarrow \ln|(u^2+2u+1)x^2| = C_2$$

$$(u^2+2u-1)x^2 = C$$

$$u=\frac{y}{x} \Rightarrow \left(\frac{y^2}{x^2}+\frac{2y}{x}-1\right)x^2 = C \Rightarrow$$

$$\boxed{y^2+2xy-y^2 = C}$$