Homework Assignment 3 in Differential Equations, MATH308

due to Feb 15, 2012

Topics covered : *exact equations; solutions of linear homogeneous equations of second order, Wronskian; linear homogeneous equations of second order with constant coefficient: the case of two distinct real roots of the characteristic polynomial (corresponds to sections 2.6, 3.2, and 3.1 in the textbook).*

1. (a) Determine whether the differential equation $(2 - 9xy^2)x + (4y^2 - 6x^3)y\dfrac{dy}{dx} = 0$ is exact. If it is exact, find the general solution;

   (b) Find the values of parameters a and b for which the differential equation

   $$(ax^2y + y^3)\,dx + (\frac{1}{3}x^3 + bxy^2)\,dy = 0$$

   is exact, and then solve it in the case of those values of a and b.

   (c) Show that the differential equation $(x^2 + y^2 + x)\,dx + y\,dy = 0$ is not exact but becomes exact when multiplied by the integrating factor $\mu(x, y) = \dfrac{1}{x^2 + y^2}$. Then solve the equation.

2. (a) Find the general solution of differential equation $y'' + y' - 6y = 0$;

   (b) Find the solution of the same equation satisfying the initial condition $y(0) = 1$, $y'(0) = \alpha$. Then find α so that the solution approaches zero as $t \to +\infty$.

   (c) Consider the differential equation

   $$y'' - 2\beta y' + (\beta^2 - 1)y = 0.$$

   (here β is a parameter). Determine the values of β, if any, for which all solutions tend to zero as $t \to \infty$; also determine the values of β, if any, for which all (nonzero) solutions become unbounded as $t \to +\infty$.

3. (a) Find the Wronskian of the functions $e^{-3t}\cos(2t)$ and $e^{-3t}\sin(2t)$;

   (b) Consider the differential equation

   $$t^2y'' - 3ty' + 3y = 0, \quad t > 0 \tag{1}$$

   Verify that $y_1(t) = t$ and $y_1(t) = t^3$ are solutions of this equation. Then prove, using the notion of Wronskian, that $y(t) = c_1t + c_2t^3$ is the general solution of this equation (on $t > 0$).

   (c) If $W(f, g)$ is the Wronskian of functions f and g and if $u = 3f - 4g$, $v = 2f + 3g$, find the Wronskian $W(u, v)$ of u and v in terms of $W(f, g)$

4. (bonus-20 points) Read the formulation and the proof of Theorem 3.2.6 on the page 153 (Abel's Theorem). Then using this theorem:

   (a) Find the Wronskian of two solutions of the equation

   $$t^2y'' + t(t - 3)y' + t^3y = 0$$

   without solving the equation;

   (b) If y_1 and y_2 are fundamental set of solutions of $ty'' - 5y' + \sin t\, y = 0$ and if $W(y_1, y_2)(2) = 2$, find the value of $W(y_1, y_2)(3)$.