MATH 308 Homework assignment 3, Solutions

Problem 1) (a) $(2-9xy^2)x+(4y^2-6x^3)y\frac{dy}{dx}=0$

$P = 2x-9x^2y^2 \qquad P_y = -18x^2y$

$Q = 4y^3-6x^3y \qquad Q_x = -18x^2y$

$\Rightarrow P_y = Q_x$ on $\mathbb{R}^2$ $\Rightarrow$ the equation is exact

Find the potential

$$\begin{cases} \varphi_x = 2x-9x^2y^2 \\ \varphi_y = 4y^3-6x^3y \end{cases} \Rightarrow \varphi = \int(2x-9x^2y^2)dx + h(y) = x^2-3x^3y^2+h(y) \Rightarrow$$

Substituting to the second equation

$\varphi_y = \cancel{-6x^3y} + h'(y) = 4y^3 \cancel{-6x^3y} \Rightarrow$

$h(y) = y^4 + C$. We can take $C=0 \Rightarrow$

$\varphi(x,y) = x^2-3x^3y^2+y^4 \Rightarrow$

General solution is $\boxed{x^2-3x^3y^2+y^4 \equiv C}$

(b) $(ax^2y+y^3)dx+(\frac{1}{3}x^3+bxy^2)dy = 0$

$P = ax^2y+y^3 \Rightarrow P_y = ax^2+3y^2$

$Q = \frac{1}{3}x^3+bxy^2 \qquad Q_x = x^2+by^2$

Eq. is exact $\Leftrightarrow$ $P_y \equiv Q_x \Leftrightarrow$ $ax^2+3y^2 = x^2+by^2$ for all x and y $\Leftrightarrow$ $\boxed{\begin{matrix} a=1 \\ b=3 \end{matrix}}$

Find the potential

$\varphi_x = x^2y+y^3 \Rightarrow \varphi = \int(x^2y+y^3)dx + h(y) = \frac{x^3}{3}y+y^3x+h(y)$

$\varphi_y = \frac{1}{3}x^3+3xy^2$

Substituting to the second equation

$\varphi_y = \cancel{\frac{x^3}{3}} + \cancel{3y^2x} + h'(y) = \cancel{\frac{1}{3}x^3} + \cancel{3xy^2} \Rightarrow$

$h'(y)=0 \Leftrightarrow h(y) \equiv C$. We can take $C=0$

$$\Rightarrow \varphi(x,y) = \frac{x^3y}{3} + y^3x \Rightarrow$$

The general solution is

$$\frac{x^3y}{3} + y^3x \equiv const$$

(c) $(x^2+y^2+x)\,dx + y\,dy = 0$

$P = x^2+y^2+x \Rightarrow P_y = 2y$

$Q = y \qquad\qquad Q_x = 0$

$\Rightarrow 2y \not\equiv 0 \Rightarrow$ the equation is not exact

With the integrating factor $\mu(x,y) = \frac{1}{x^2+y^2}$ we get the equation

$$\frac{1}{x^2+y^2}(x^2+y^2+x)\,dx + \frac{y}{x^2+y^2}\,dy = 0 \Leftrightarrow$$

$$\underbrace{\left(1 + \frac{x}{x^2+y^2}\right)}_{\tilde{P}} dx + \underbrace{\frac{y}{x^2+y^2}}_{\tilde{Q}}\,dy = 0$$

$\tilde{P} = 1 + \frac{x}{x^2+y^2} \Rightarrow \tilde{P}_y = -\frac{2yx}{(x^2+y^2)^2}$

$\tilde{Q} = \frac{y}{x^2+y^2} \Rightarrow \tilde{Q}_x = -\frac{2xy}{(x^2+y^2)^2}$

$\Rightarrow \tilde{P}_y \equiv \tilde{Q}_x$

Actually here the equality holds on $\mathbb{R}^2 \setminus$ origin ~~not~~ simply connected (so we cannot immediately conclude that the equation is exact)

Let us try to find the potential.

$$\begin{cases} \varphi_x = 1 + \frac{x}{x^2+y^2} \Rightarrow \varphi = \int\left(1+\frac{x}{x^2+y^2}\right)dx + h(y) = x + \frac{1}{2}\ln(x^2+y^2) + h(y) \Rightarrow \\ \varphi_y = \frac{y}{x^2+y^2} \end{cases}$$

$$\varphi_y = \frac{y}{x^2+y^2} + h'(y) = \frac{y}{x^2+y^2} \Rightarrow h'(y) = 0 \Rightarrow$$

$h(y) = const$; We can take $h(y) = 0 \Rightarrow$

The potential exists and can be taken as

as $\varphi(x,y) = \frac{1}{2}\ln|x^2+y^2) + x$ => the equation is exact.

General solution is $\boxed{\frac{1}{2}\ln|x^2+y^2) + x = const}$

Problem (2)

(a) $y'' + y' - 6y = 0$

The characteristic equation is

$$r^2 + r - 6 = 0$$

$$D = 1 + 4\cdot 6 = 1 + 24 = 25$$

$$r_1 = \frac{-1+5}{2} = 2$$

$$r_2 = \frac{-1-5}{2} = -3$$

$\Downarrow$

The general solution is

$$y(t) = C_1 e^{2t} + C_2 e^{-3t}$$

(b) $y(0) = 1 \Leftrightarrow C_1 + C_2 = 1$

$y'(t) = 2C_1 e^{2t} - 3C_2 e^{-3t}$ =>

$y'(0) = 2C_1 - 3C_2 = \alpha$

We get the following system of equations for C_1 and C_2:

$$\begin{cases} C_1 + C_2 = 1 \\ 2C_1 - 3C_2 = \alpha \end{cases}$$

Eliminate C_1

(II) - 2(I)

$-3C_2 - 2C_2 = \alpha - 2$ => $-5C_2 = \alpha - 2$

$$C_2 = \frac{2-\alpha}{5}$$

$$C_1 = 1 - C_2 = 1 - \frac{2-\alpha}{5} = \frac{3+\alpha}{5} \Rightarrow$$

$$\boxed{y(t) = \frac{\alpha+3}{5} e^{2t} + \frac{2-\alpha}{5} e^{-3t}}$$

$y(t) \underset{t\to+\infty}{\longrightarrow} 0 \Leftrightarrow$ the coefficient near e^{2t} vanishes $\Leftrightarrow$

$$\alpha + 3 = 0 \Leftrightarrow \boxed{\alpha = -3}$$

(c) $y'' - 2\beta y' + (\beta^2 - 1)y = 0$

The characteristic equation is

$$r^2 - 2\beta r + (\beta^2 - 1) = 0$$

Way 1 Quadratic formula

$$D = 4\beta^2 - 4(\beta^2 - 1) = \cancel{4\beta^2} - \cancel{4\beta^2} + 4 = 4$$

$$r_1 = \frac{2\beta + 2}{2} = \beta + 1$$

$$r_2 = \frac{2\beta - 2}{2} = \beta - 1$$

Way 2 Vieta Theorem: $r_1 r_2 = \beta^2 - 1$, $r_1 + r_2 = 2\beta$ $\Rightarrow$ one can guess that $r_1 = \beta + 1$, $r_2 = \beta - 1$

General solution is

$$y(t) = C_1 e^{(\beta+1)t} + C_2 e^{(\beta-1)t}$$

All solutions tend to zero as $t \to +\infty \Leftrightarrow$ both roots of the characteristic equation are negative $\Leftrightarrow \begin{cases} \beta - 1 < 0 \\ \beta + 1 < 0 \end{cases} \Leftrightarrow$

$$\begin{cases} \beta < 1 \\ \beta < -1 \end{cases} \Leftrightarrow \boxed{\beta < -1}$$

All (nonzero) solutions become unbounded as $t \to +\infty \Leftrightarrow$ both roots of characteristic equation are positive $\Leftrightarrow$

$$\begin{cases} \beta - 1 > 0 \\ \beta + 1 > 0 \end{cases} \Leftrightarrow \begin{cases} \beta > 1 \\ \beta > -1 \end{cases} \Leftrightarrow \boxed{\beta > 1}$$

Problem 3

a) $(e^{-3t}\cos 2t)' = -3e^{-3t}\cos 2t - 2e^{-3t}\sin 2t$

$(e^{-3t}\sin 2t)' = -3e^{-3t}\sin 2t + 2e^{-3t}\cos 2t$

$\Downarrow$

$$\text{Wronskian} = \begin{vmatrix} e^{-3t}\cos 2t & e^{-3t}\sin 2t \\ -3e^{-3t}\cos 2t - 2e^{-3t}\sin 2t & -3e^{-3t}\sin 2t + 2e^{-3t}\cos 2t \end{vmatrix} = (*)$$

1) Way one Using the property of determinants: a determinant does not change if we add $3 \times$ the first row to the second row:

$$(*) = \begin{vmatrix} e^{-3t}\cos 2t & e^{-3t}\sin 2t \\ -2e^{-3t}\sin 2t & 2e^{-3t}\cos 2t \end{vmatrix} = 2e^{-6t}\big(\underbrace{\cos^2 2t + \sin^2 2t}_{1}\big) =$$

$$= \boxed{2e^{-6t}}$$

2) Way two You can obtain the same by direct calculations using the definition of the determinant

$$(*) = e^{-3t}\cos 2t\left(-3e^{-3t}\sin 2t + 2e^{-3t}\cos 2t\right) - e^{-3t}\sin 2t\left(-3e^{-3t}\cos 2t - 2e^{-3t}\sin 2t\right) = -3e^{-6t}\cancel{\cos 2t \sin 2t} + 2e^{-6t}\cos^2 2t + 3e^{-6t}\cancel{\sin 2t \cos 2t} + 2e^{-6t}\sin^2 2t = \boxed{2e^{-6t}}$$

Problem 3 b) $t^2y'' - 3ty' + 3y = 0$

1) Verify that $y_1(t) = t$ is a solution by substituting it into the equation: $y_1' = 1$, $y_1'' = 0 \Rightarrow$

$$t^2 \cdot 0 - 3t \cdot 1 + 3t = -3t + 3t \Rightarrow y_1(t) \text{ is a solution.}$$

2) Verify that $y_2(t) = t^3$ is a solution by substituting it into the equation: $y_2'(t) = 3t^2$

$$y_2''(t) = 6t$$

$$t^2 6t - 3t \cdot 3t^2 + 3t^3 = 6t^3 - 9t^3 + 3t^3 = 0 \Rightarrow$$

$y_2(t)$ is a solution

3) Check that $\{y_1(t), y_2(t)\}$ is a fundamental system on $t>0$ by showing that the Wronskian of these two functions is $\neq 0$

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} t & t^3 \\ 1 & 3t^2 \end{vmatrix} = 3t^3 - t^3 =$$

$$2t^3 \neq 0 \text{ on } t>0$$

Problem 3 c) $u' = 3f' - 4g'$
$v' = 2f' + 3g'$

$$W(u,v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} 3f-4g & 2f+3g \\ 3f'-4g' & 2f'+3g' \end{vmatrix} =$$

$$= (3f-4g)(2f'+3g') - (2f+3g)(3f'-4g') =$$

$$= \cancel{6ff'} - \underline{8gf'} + \underline{\underline{9fg'}} - \cancel{12gg'} - \cancel{6ff'} -$$

$$- \underline{9f'g} + \underline{\underline{8fg'}} + \cancel{12gg'} = 17(fg' - f'g) =$$

$$= 17 \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = 17\, W(f,g) \Rightarrow$$

$$\boxed{W(u,v) = 17\, W(f,g)}$$

Remark In general if $u = af + bg$, $v = cf + dg$ for some constants a, b, c, d then

$$W(u,v) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} W(f,g)$$

In our case $a = 3$, $b = -4$, $c = 2$, $d = 3$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 3 & -4 \\ 2 & 3 \end{vmatrix} = 9 - (-8) = 17$$

Problem 4 Abel's theorem says that if $y_1(t)$ and $y_2(t)$ are solutions of equation

$$y'' + p(t)y' + q(t)y = 0$$ then their Wronskian

W satisfies the differential equation of the first order

$$W' = -pW \Leftrightarrow W = Ce^{\int p(t)dt}$$

Then a) $t^2y'' + t(t-3)y' + t\,y = 0 \Leftrightarrow$ (dividing by t^2)

$$y^2 + \frac{t-3}{t}y' + ty = 0 \Rightarrow p(t) = \frac{t-3}{t} = 1 - \frac{3}{t} \Rightarrow$$

By Abel's theorem

$$W = Ce^{-\int p\,dt} = Ce^{-\int 1-\frac{3}{t}\,dt} = Ce^{\int\left(\frac{3}{t}-1\right)dt} =$$

$$= Ce^{3\ln|t|-t} = \boxed{C|t|^3e^{-t}}$$

(8) $ty'' - 5y' + \sin t\, y = 0$

Divide by t:

$y'' - \frac{5}{t}y' + \frac{\sin t}{t}y = 0 \Rightarrow p(t) = -\frac{5}{t} \Rightarrow$ (By Abel's Theorem)

$W = Ce^{-\int p\,dt} = Ce^{\int\frac{5}{t}dt} = Ce^{5\ln t} = Ct^5$ for $t>0$

If $W(2) = 2$, then $2 = C2^5 \Rightarrow C = \frac{1}{2^4} = \frac{1}{16} \Rightarrow$

$W(3) = C\cdot 3^5 = \frac{1}{16}\cdot 3^5 = \boxed{\frac{243}{16}}$