MATH 308 Homework Assignment 4 Solutions

1) a) $e^{\frac{3\pi}{4}i} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = \boxed{-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}}$

b) $e^{\left(4-\frac{\pi}{3}i\right)} = e^4\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right) = e^4\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) =$

$= \frac{e^4}{2} - i\frac{e^4\sqrt{3}}{2}$

2) $y'' - 2y' + 10y = 0$

(a) Characteristic equation is

$r^2 - 2r + 10 = 0$

$D = 4 - 40 = -36$ | or $D/4 = 1 - 10 = -9$

$r_{1,2} = \frac{2 \pm \sqrt{-36}}{2} = \frac{2 \pm 6i}{2} = 1 \pm 3i$ | $r_{1,2} = 1 \pm \sqrt{-9} = 1 \pm 3i$

$\lambda = 1$ and $\mu = 3 \Rightarrow$

$e^t\cos 3t$ and $e^t\sin 3t$ form a fundamendal set of solution $\Rightarrow$

The general solution is

$$\boxed{C_1 e^t\cos 3t + C_2 e^t \sin 3t}$$

(b) $y\left(\frac{\pi}{2}\right) = 5 \Leftrightarrow C_1 e^{\frac{\pi}{2}}\underbrace{\cos\frac{3\pi}{2}}_{0} + C_2 e^{\frac{\pi}{2}}\underbrace{\sin\frac{3\pi}{2}}_{-1} = -C_2 e^{\frac{\pi}{2}} = 5 \Rightarrow$

$$\boxed{C_2 = -5e^{-\pi/2}}$$

$y'(t) = -3C_1 e^t\sin 3t + C_1 e^t\cos 3t + 3C_2 e^t\cos 3t + C_2 e^t\sin 3t \Rightarrow$

$y'\left(\frac{\pi}{2}\right) = -1 \Leftrightarrow y'\left(\frac{\pi}{2}\right) = -3C_1 e^{\frac{\pi}{2}}\underbrace{\sin\frac{3\pi}{2}}_{-1} + C_1 e^{\frac{\pi}{2}}\underbrace{\cos\frac{3\pi}{2}}_{0} + 3C_2 e^{\pi/2}\underbrace{\cos\frac{3\pi}{2}}_{0} + \underbrace{C_2}_{-5e^{-\pi/2}} e^{\frac{\pi}{2}}\underbrace{\sin\frac{3\pi}{2}}_{(-1)} =$

$$= 3C_1 e^{\frac{\pi}{2}} + 5\underbrace{e^{-\frac{\pi}{2}}e^{\frac{\pi}{2}}}_{1} = 3C_1 e^{\frac{\pi}{2}} + 5 = -1 \Rightarrow$$

$$3C_1 e^{\frac{\pi}{2}} = -6 \Rightarrow \boxed{C_1 = -2e^{-\pi/2}} \Rightarrow$$

$$y(t) = e^{t}e^{-\frac{\pi}{2}}(-2\cos 3t - 5\sin 3t) = \boxed{e^{t-\frac{\pi}{2}}(-2\cos 3t - 5\sin 3t)}$$

Sketch the graph

$y(t) = e^{t-\frac{\pi}{2}} R \cos(3t-\delta)$ where

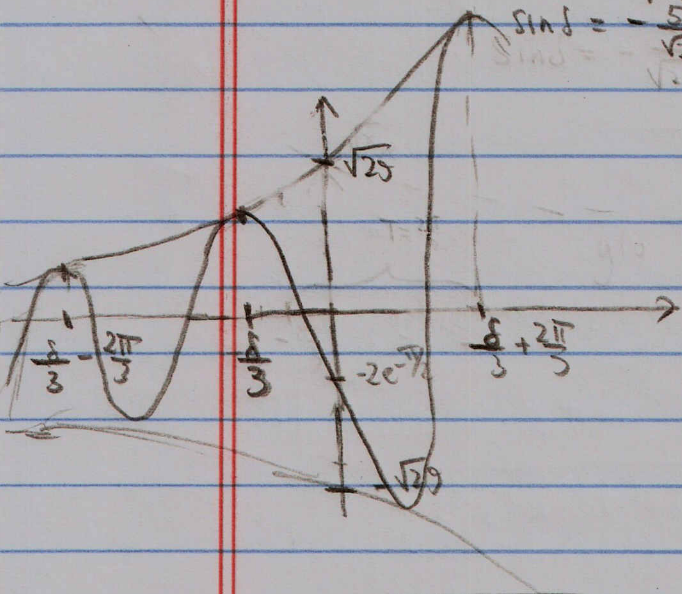
$$R = \sqrt{2^2+5^2} = \sqrt{29}$$

$\cos\delta = -\frac{2}{\sqrt{29}}$, $\sin\delta = -\frac{5}{\sqrt{29}}$ $\Rightarrow$ δ is the third quadrant

$$\delta = \arccos\frac{2}{\sqrt{29}} - \pi = \arctan\frac{5}{2} - \pi$$

$$y(0) = e^{-\frac{\pi}{2}} R\cos\delta = -2e^{-\frac{\pi}{2}}$$

The period $= \frac{2\pi}{3}$



$$\boxed{y(t) \underset{t\to-\infty}{\longrightarrow} 0}$$, because

$$|y(t)| = Re^{t-\frac{\pi}{2}} \underbrace{|\cos(3t-\delta)|}_{<1} < Re^{t-\frac{\pi}{2}} \underset{t\to-\infty}{\longrightarrow} 0$$

3) $y'' - 10y' + 25y = 0$

a) Char. equation:

$$r^2 - 10r + 25 = 0$$
$$D = 100 - 100 = 0 \rightarrow \text{repeated root}$$
$$r_{1,2} = \frac{10}{2} = 5$$

or $(r-5)^2 = 0$ - perfect square

$\Downarrow$

$r_{1,2} = 5$

$\Downarrow$

e^{5t} and te^{5t} is a fundemendal set of solutions $\Rightarrow$

The general solution is

$$C_1 e^{5t} + C_2 t e^{5t}.$$

b) $y(0) = 3 \Rightarrow C_1 e^0 + C_2 \cdot 0\, e^0 = C_1 = 3 \Rightarrow \boxed{C_1 = 3}$

$y'(t) = C_1 e^{5t} + C_2 e^{5t} + 5C_2 t e^{5t} \Rightarrow$

$y'(0) = C_1 + C_2 = 3 + C_2 = \alpha \Rightarrow \boxed{C_2 = \alpha - 3}$

$\Downarrow$

$$y(t) = 3e^{5t} + (\alpha - 3)te^{5t}$$

c) $y(t) = (3 + (\alpha-3)t)e^{5t}$

Everything depends on the sign of $\alpha - 3$:

If $\alpha - 3 > 0 \Rightarrow y(t) = (3 + (\alpha-3)t)e^{5t} \xrightarrow[t \to +\infty]{} +\infty \cdot +\infty = +\infty$

If $\alpha - 3 = 0 \Rightarrow y(t) = 3e^{5t} \xrightarrow[t \to +\infty]{} +\infty$

If $\alpha - 3 < 0 \Rightarrow y(t) = (3 + (\alpha-3)t)e^{5t} \rightarrow -\infty \cdot +\infty = -\infty$

Conclusion: $y(t) \rightarrow +\infty \Leftrightarrow \alpha \geq 3$

$y(t) \rightarrow -\infty \Leftrightarrow \alpha < 3$

4. Following the method of reduction of order we find a solution in the form

$$y(t) = v(t)\cdot t^{-1} = \frac{v}{t}$$

Find the equation for v substituting into equation:

$$\begin{array}{r|l} -5 & y(t) = \dfrac{v}{t} \\ -3t\times & y'(t) = -\dfrac{v}{t^2} + \dfrac{v'}{t} \\ t^2\times & y''(t) = \dfrac{2v}{t^3} - \dfrac{v'}{t^2} - \dfrac{v'}{t^2} + \dfrac{v''}{t} = \\ & = \dfrac{2v}{t^3} - \dfrac{2v'}{t^2} + \dfrac{v''}{t} \end{array}$$

$$t^2y'' - 3ty' - 5y = \underbrace{\left(\frac{2}{t} + \frac{3}{t} - \frac{5}{t}\right)}_{0} v + (-2-3)v' + tv'' = 0 \Rightarrow$$

$$tv'' - 5v' = 0$$

Substitute $w = v'$

$$tw' - 5w = 0 \Rightarrow w' = \frac{5}{t}w \Rightarrow w = \tilde{C}_1 e^{\int \frac{5}{t}dt} = \tilde{C}_1 e^{5\ln t} = \tilde{C}_1 t^5 \Rightarrow$$

$$w = \tilde{C}_1 t^5 \Rightarrow \quad v' = \tilde{C}_1 t^5 \Rightarrow v = \int \tilde{C}_1 t^5 dt + C_2 = \underbrace{\frac{\tilde{C}_1}{6}}_{C_1} t^6 + C_2 \Rightarrow$$

$$y = \frac{v}{t} = C_1 t^5 + \frac{C_2}{t} \Rightarrow \boxed{y(t) = C_1 t^5 + \frac{C_2}{t}} \quad (*)$$

So as an additional solution one can take $\boxed{y_2(t) = t^5}$

(of course one can take any other solution of the form (*) which is not constant multiple of t^{-1})

5a) The characteristic equation is

$$r^2 + \omega_0^2 = 0 \Rightarrow r^2 = -\omega_0^2 \Rightarrow r = \pm i\omega_0$$

Case 1 $\omega^2 \neq \omega_0^2$ $(\omega \neq 0, \omega_0 \neq 0)$ Then ωi is not a root of the characteristic polynomial $\Rightarrow$ (i.e. its multiplicity $s = 0$)

$\Rightarrow$ We look for a particular solution in the form

$$y(t) = A_1 \cos\omega t + A_2 \sin\omega t$$

Plug in this $y(t)$ into equation:

$$y'(t) = \omega A_2 \cos\omega t - \omega A_1 \sin\omega t$$

$$y''(t) = -\omega^2 A_1 \cos\omega t - \omega^2 A_2 \sin\omega t$$

$\Downarrow$

$$y'' + \omega_0^2 y = (-\omega^2 A_1 + \omega_0^2 A_1)\cos\omega t + (-\omega^2 A_2 + \omega_0^2 A_2)\sin\omega t = \sin\omega t \Rightarrow$$

$$A_1(\omega_0^2 - \omega^2) = 0 \Rightarrow A_1 = 0$$

$$A_2(\omega_0^2 - \omega^2) = 1 \Rightarrow A_2 = \frac{1}{\omega_0^2 - \omega^2}$$ $\Rightarrow$ a particular solution

can be taken as $$y_p(t) = \frac{1}{\omega_0^2 - \omega^2} \sin\omega t \Rightarrow$$

The general solution is

$$\boxed{y(t) = \frac{1}{\omega_0^2 - \omega^2} \sin\omega t + \underbrace{C_1 \cos\omega_0 t + C_2 \sin\omega_0 t}_{\text{gen. solution of hom. eq.}}}$$

Case 2 If $\omega^2 = \omega_0^2 \neq 0$, then ωi is a root (not repeated) of the characteristic equation, i.e $s = 1$

We look for a particular solution in the form

$$y(t) = t(A_1 \cos\omega t + A_2 \sin\omega t)$$

Plug in this $y(t)$ into equation

$$y' = A_1 \cos\omega t + A_2 \sin\omega t + t\left(-A_1\omega\sin\omega t + A_2\omega\cos\omega t\right)$$

$$y'' = -A_1\omega\sin\omega t + A_2\omega\cos\omega t - A_1\omega\sin\omega t + A_2\omega\cos\omega t +$$

$$+ t\left(-A_1\omega^2\cos\omega t - A_2\omega^2\sin\omega t\right) = -2A_1\omega\sin\omega t + 2A_2\omega\cos\omega t -$$

$$- t\omega^2\left(A_1\cos\omega t + A_2\sin\omega t\right)$$

$$\Rightarrow y'' + \omega_0^2 y \underset{\substack{\downarrow \\ \text{here use that } \omega^2 = \omega_0^2}}{=} \cancel{-t\omega^2\left(A_1\cos\omega t + A_2\sin\omega t\right)} - 2\omega A_1\sin\omega t + 2A_2\omega\cos\omega t +$$

$$\cancel{+ t\omega_0^2\left(A_1\cos\omega t + A_2\sin\omega t\right)} = -2\omega A_1\sin\omega t + 2\omega A_2\cos\omega t = \sin\omega t$$

$$\Downarrow$$

$$-2\omega A_1 = 1 \Rightarrow A_1 = -\frac{1}{2\omega}$$

$$2\omega A_2 = 0 \Rightarrow A_2 = 0$$

$\Downarrow$ a particular solution can be taken in the form

$$y_p(t) = -\frac{t}{2\omega}\cos\omega t \quad \Rightarrow$$

The general solution is

$$y(t) = -\frac{t}{2\omega}\cos\omega t + C_1\cos\omega_0 t + C_2\sin\omega_0 t$$

(one can write also $y(t) = -\frac{t}{2\omega_0}\cos\omega_0 t + C_1\cos\omega_0 t + C_2\sin\omega_0 t$)

<u>Case 3</u> Formally we also have to consider the cases when one or both ω and ω_0 are zeros. This cases are not so interesting and the points will not be taken off if you did not consider them (I probably had to indicate in the problem formulation that both ω_0 and ω_1 are not zero)

<u>Important</u> <u>Remark</u> The main difference between case 1 and case 2 is that in case 1 any solution is bounded and in case 2 any solution is unbounded. The first case is called non-resonant, because the frequency ω of the external force (the right-hand side of the equation) is not equal to the natural frequency ω_0 of the system and the second case is called resonant because the frequency of the external force is equal to the natural frequency. The resonance leads to increasing the amplitude and may cause the destruction of the system (this will be also discussed in section 3.8).

5 b) $y'' - 3y' + 2y = 5e^{2t} + e^{3t}\cos 2t$

We find separately particular solutions of

$$y'' - 3y' + 2y = 5e^{2t} \quad \text{and}$$

$$y'' - 3y' + 2y = e^{3t}\cos 2t$$

The characteristic equation is

$$r^2 - 3r + 2 = 0$$

$$D = 9 - 8 = 1$$

$$r_1 = \frac{3+1}{2} = 2, \quad r_2 = \frac{3-1}{2} = 1$$

The roots are 1 and 2

<u>Case 1</u>: $y'' - 3y' + 2y = 5e^{2t}$ (1)

Since $\alpha = 2$ is a root (but not a repeated root) of the characteristic equation we look for a particular solution in the form

$$y(t) = Ate^{2t} \qquad (\text{i.e the multiplicity } s = 1)$$

Plug into equation

$$y'(t) = 2Ate^{2t} + Ae^{2t}$$

$$y''(t) = 4Ate^{2t} + 2Ae^{2t} + 2Ae^{2t} = 4Ate^{2t} + 4Ae^{2t}$$

$$\Downarrow$$

$$y'' - 3y' + 2y = \underbrace{(4A - 6A + 2A)}_{=0} te^{2t} + (4A - 3A + 2A)e^{2t} =$$

$$= 3Ae^{2t} = 5e^{2t} \Rightarrow 3A = 5 \Rightarrow A = \frac{5}{3} \Rightarrow$$

a particular solution of (1) can be taken in the form

$$Y_{1_p} = \frac{5}{3}te^{2t}$$

<u>Case 2</u>: $y'' - 3y' + 2y = e^{3t}\cos 2t$

$\alpha = 2, \beta = 3$. $2 + 3i$ is not a root of characteristic polynomial $\Rightarrow$ a particular solution can be found in the form $y(t) = A_1 e^{3t}\cos 2t + A_2 e^{3t}\sin 2t$

Plug into equation

$$y(t) = A_1 e^{3t}\cos 2t + A_2 e^{3t}\sin 2t$$

$$y'(t) = 3A_1 e^{3t}\cos 2t - 2A_1 e^{3t}\sin 2t + 3A_2 e^{3t}\sin 2t +$$
$$+ 2A_2 e^{3t}\cos 2t = (3A_1 + 2A_2)e^{3t}\cos 2t + (3A_2 - 2A_1)e^{3t}\sin 2t$$

$$y''(t) = (9A_1 + 6A_2)e^{3t}\cos 2t - (6A_1 + 4A_2)e^{3t}\sin 2t + (9A_2 - 6A_1)e^{3t}\sin 2t +$$
$$+ (6A_2 - 4A_1)e^{3t}\cos 2t = (5A_1 + 12A_2)e^{3t}\cos 2t + (-12A_1 + 5A_2)e^{3t}\sin 2t \Rightarrow$$

$$\Rightarrow$$

$$y'' - 3y' + 2y = (5A_1 + 12A_2 - 9A_1 - 6A_2 + 2A_1)e^{3t}\cos 2t +$$
$$+ (-12A_1 + 5A_2 - 9A_2 + 6A_1 + 2A_2)e^{3t}\sin 2t =$$
$$= (-2A_1 + 6A_2)e^{3t}\cos 2t + (-6A_1 - 2A_2)e^{3t}\sin 2t = e^{3t}\cos 2t$$

Comparing coefficients:

$$\begin{cases} -2A_1 + 6A_2 = 1 \\ -6A_1 - 2A_2 = 0 \end{cases} \Rightarrow$$

Eliminate A_2: $I + 3\times II = -2A_1 - 18A_1 = 1 \Rightarrow$

$$-20A_1 = 1 \Rightarrow \boxed{A_1 = -\frac{1}{20}} \Rightarrow$$

$$6A_2 = 1 + 2A_1 = 1 - \frac{1}{10} = \frac{9}{10} \Rightarrow$$

$$A_2 = \frac{9}{6\cdot 10} = \frac{3}{20} \Rightarrow A_2 = \frac{3}{20}$$

$$\Downarrow$$

A particular solution of (2) can be taken as

$$Y_{2p} = -\frac{1}{20}e^{3t}\cos 2t + \frac{3}{20}e^{3t}\sin 2t$$

Then a particular solution of the original equation can be taken as $Y_{1p}(t)+Y_{2p}(t) = \frac{5}{3}e^{2t} - \frac{1}{20}e^{3t}\cos 2t + \frac{3}{20}e^{3t}\sin 2t$

$\Downarrow$

The general solution of the original non-homogeneous equation = a particular solution + the general solution of hom. equation =

$$= \boxed{\frac{5}{3}e^{2t} - \frac{1}{20}e^{3t}\cos 2t + \frac{3}{20}e^{3t}\sin 2t + C_1 e^{2t} + C_2 e^t}$$

(recall that the roots of char. equation are 1 and 2)

6) a) Consider 2 cases:

<u>Case 1</u>: Roots are 2 distinct real number r_1 and r_2

Then any solution have a form

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} = e^{r_1 t}\left(C_1 + C_2 e^{(r_2-r_1)t}\right) \Rightarrow$$

If $C_1 = C_2 = 0$ then $y(t) \equiv 0$. Otherwise,

$y(t) = 0 \Leftrightarrow C_1 + C_2 e^{(r_2-r_1)t} = 0 \Leftrightarrow$ either $C_2 = 0 \Rightarrow$ no zero values, because $C_1 \neq 0$

or $C_2 \neq 0$ and $e^{(r_1-r_2)t} = -\frac{C_1}{C_2} \Rightarrow$

If $\frac{C_1}{C_2} \geq 0 \Rightarrow -\frac{C_1}{C_2} \leq 0 \Rightarrow$ no zero values

If $\frac{C_1}{C_2} < 0 \Rightarrow t = \frac{1}{r_1 - r_2}\ln\left(-\frac{C_1}{C_2}\right) \Rightarrow$ one t taking on the value zero

$\Rightarrow$ at most one t taking on the value zero

<u>Case 2</u> One repeated root r.

Then any solution have a form

$y(t) = C_1 e^{rt} + C_2 t e^{rt} = (C_1 + C_2 t)e^{rt} \Rightarrow$ If $C_1 = C_2 = 0 \Rightarrow y(t) \equiv 0$.

Otherwise, $y(t) = 0 \Leftrightarrow C_1 + C_2 t = 0 \Rightarrow$ either $C_2 = 0 \rightarrow$ no zero values because $C_1 \neq 0$

or $t = -\frac{C_1}{C_2}$, i.e. there is one t taking on the value zero $\Rightarrow$

$\Rightarrow$ at most one t taking on the value zero.

So in both cases we got the required conclusion.

b) In this case any solution have the form

$y(t) = Re^{\lambda t}\cos(\mu t - \delta)$. If $R=0$ then $y(t)\equiv 0$ - infinite many t

$$y(t)=0 \Leftrightarrow \cos(\mu t-\delta)=0 \Leftrightarrow$$

$$\mu t - \delta = \frac{\pi}{2}+\pi n,\ n\in\mathbb{Z}$$

$$\mu t = \delta + \frac{\pi}{2}+\pi n,\ n\in\mathbb{Z} \Rightarrow$$

$$t = \frac{1}{\mu}\left(\delta+\frac{\pi}{2}+\pi n\right),\ n\in\mathbb{Z} \Rightarrow$$

infinite many t taking on the value zero.

c) Consider 2 cases

Case 1 The roots r_1 and r_2 of the characteristic polynomial are real $\Rightarrow$ Both of them has to be negative, Because from the assumptions that a, b, and c are positive it follows that $ar^2+br+c>0$ for $r\geq 0$ $\Rightarrow$ no nonnegative roots of char. eq.

The general solution is either $c_1e^{r_1t}+c_2e^{r_2t}$ if $r_1\neq r_2$ or $c_1e^{r_1t}+c_2te^{r_1t}$ (if $r_1=r_2$). In both cases it approach 0 as $t\to+\infty$ because r_1 and r_2 are negative.

Case 2 If the roots of characteristic polynomial are complex, $r_{1,2}=\lambda\pm i\mu \Rightarrow$ by Viete Thm $r_1+r_2=-\frac{b}{a}<0$ but

$r_1+r_2=2\lambda \Rightarrow 2\lambda=-\frac{b}{2a}<0 \Rightarrow \lambda<0$

The general solution is

$$Re^{\lambda t}\cos(\mu t-\delta) \xrightarrow[t\to+\infty]{} 0 \quad \text{because } \lambda<0.$$