MATH 308 Homework 7 Solutions

Problem 1

$$y'' + 4y = g(t)$$
$$y(0) = 1$$
$$y'(0) = 3$$

$$g(t) = \begin{cases} \sin t, & 0 \leq t < 2\pi \\ 0, & t \geq 2\pi \end{cases}$$

Solution

Left-hand side: $g(t) = \sin t - \sin t \, u_{2\pi}(t) = \sin t - \sin(t-2\pi) u_{2\pi}(t)$

$\Downarrow$

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{\sin t\} - \underbrace{\mathcal{L}\{\sin(t-2\pi) u_{2\pi}(t)\}}_{\text{translation in } t \text{ property}} = \frac{1}{s^2+1} + \frac{e^{-2\pi s}}{s^2+1}$$

Right-hand side: $\mathcal{L}\{y\} = Y(s)$

$$\mathcal{L}\{y''\} = s^2 Y(s) - s\,y(0) - y'(0) = s^2 Y(s) - s - 3 \Rightarrow$$

$$\mathcal{L}\{y'' + 4y\} = (s^2+4)Y(s) - s - 3 = \frac{1}{s^2+1} - \frac{e^{-2\pi s}}{s^2+1} \Rightarrow$$

$$Y(s) = \frac{1}{(s^2+1)(s^2+4)} + \frac{s+3}{s^2+4} - \frac{e^{-2\pi s}}{(s^2+1)(s^2+4)} \quad (*)$$

i) Let $F(s) = \frac{1}{(s^2+1)(s^2+4)}$ Find $\mathcal{L}^{-1}\{F(s)\}$

Set $x = s^2$ $\quad \frac{1}{(x+1)(x+4)} = \frac{A}{x+1} + \frac{B}{x+4} \Rightarrow$

$$1 = A(x+4) + B(x+1)$$
$$x = -1 \Rightarrow 1 = 3A \Rightarrow A = \frac{1}{3}$$
$$x = -4 \Rightarrow 1 = -3B \Rightarrow B = -\frac{1}{3}$$

$$\Rightarrow \frac{1}{(x+1)(x+4)} = \frac{1}{3}\left(\frac{1}{x+1} - \frac{1}{x+4}\right) \Rightarrow F(s) = \frac{1}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right) \Rightarrow$$

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \frac{1}{3}\cdot\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t$$

(ii) $$\mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{(s^2+1)(s^2+4)}\right\} \underset{\substack{\text{Translation} \\ \text{in } t \text{ propert}}}{=} u_{2\pi}(t)\left(\frac{1}{3}\sin(t-2\pi)-\frac{1}{6}\sin 2(t-2\pi)\right)=$$

$$= u_{2\pi}(t)\left(\frac{1}{3}\sin t-\frac{1}{6}\sin 2t\right)$$

(iii) $$\frac{s+3}{s^2+4} = \frac{s}{s^2+4}+\frac{3}{2}\,\frac{2}{s^2+4} \Rightarrow$$

$$\mathcal{L}^{-1}\left\{\frac{s+3}{s^2+4}\right\} = \cos 2t+\frac{3}{2}\sin 2t$$

$\Downarrow$ Combining (i), (ii) & (iii) and using (*)

$$y(t) = \boxed{(1-u_{2\pi}(t))\left(\frac{1}{3}\sin t-\frac{1}{6}\sin 2t\right)+\cos 2t+\frac{3}{2}\sin 2t} =$$

$$= \cos 2t + \frac{1}{3}(1-u_{2\pi}(t))\sin t+\frac{1}{6}\left(-1+9+u_{2\pi}(t)\right)\sin 2t$$

$$= \boxed{\cos 2t + \frac{1}{3}(1-u_{2\pi}(t))\sin t+\frac{1}{6}\left(8+u_{2\pi}(t)\right)\sin 2t}$$

Problem 2

$$y''+5y'+6y=g(t)$$
$$y(0)=0,\ y'(0)=2$$

$$g(t)=\begin{cases}0 & t<1\\ t & 1\le t<5\\ 1 & t\ge 5\end{cases}$$

Solution

Left-hand side: $g(t) = t\,u_1(t)+(1-t)\,u_5(t)$

Let us represent it in the form convinient for using the translation in t property:

(i) $$t\,u_1(t) = (t-1)u_1(t)+u_1(t) \Rightarrow$$

$$\mathcal{L}\{t\,u_1(t)\} = \mathcal{L}\{\underbrace{(t-1)}_{f(t)=t\ \ \mathcal{L}\{t\}=\frac{1}{s^2}}u_1(t)\}+\mathcal{L}\{u_1(t)\} = \frac{e^{-s}}{s^2}+\frac{e^{-s}}{s} = e^{-s}\,\frac{s+1}{s^2}$$

(ii) $(1-t)u_5(t)$. We want to represent it as

$$(1-t)u_5(t) = f_1(t-5)\,u_5(t) \Leftrightarrow f_1(t-5) = 1-t \Rightarrow$$

$$f_1(t) = f_1((t+5)-5) = 1-(t+5) = -4-t$$

$$\Downarrow$$

$$F_1(s) = -\frac{4}{s} - \frac{1}{s^2} \Rightarrow$$

$$\mathcal{L}\{(1-t)u_5(t)\} = -e^{-5s}\left(\frac{4}{s}+\frac{1}{s^2}\right) = -e^{-5s}\,\frac{4s+1}{s^2}$$

Righ-hand side:

$$\mathcal{L}\{y(t)\} = Y(s)$$

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s)$$

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2$$

$$\mathcal{L}\{y''+5y'+6y\} = (s^2+5s+6)Y(s) - 2 \Rightarrow$$

$$(s^2+5s+6)Y(s) - 2 = e^{-s}\,\frac{s+1}{s^2} - e^{-5s}\,\frac{4s+1}{s^2}$$

$$\Downarrow$$

$$Y(s) = \frac{2}{s^2+5s+6} + e^{-s}\,\frac{s+1}{s^2(s^2+5s+6)} - e^{-5s}\,\frac{4s+1}{s^2(s^2+5s+6)}$$

Note that $s^2+5s+6 = (s+2)(s+3)$. Find the inverse Laplace for all 3 terms separately

(i) $\mathcal{L}^{-1}\left\{\frac{2}{s^2+5s+6}\right\}$?

$$\frac{2}{s^2+5s+6} = \frac{2}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$$

$$2 = A(s+3) + B(s+2)$$

$$\left.\begin{array}{l} s=-2 \Rightarrow 2 = A \Leftrightarrow A = 2 \\ s=-3 \Rightarrow 2 = -B \Rightarrow B = -2 \end{array}\right\} \Rightarrow \frac{2}{(s+2)(s+3)} = 2\left(\frac{1}{s+2} - \frac{1}{s+3}\right) \Rightarrow$$

$$\mathcal{L}^{-1}\left\{\frac{2}{s^2+5s+6}\right\} = 2\left(e^{-2t}-e^{-3t}\right)$$

(ii) $\mathcal{L}^{-1}\left\{e^{-s}\frac{s+1}{s^2(s+2)(s+3)}\right\}$?

Find $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2(s+2)(s+3)}\right\}$ first

$$\frac{s+1}{s^2(s+2)(s+3)} = \frac{C}{s}+\frac{D}{s^2}+\frac{E}{s+2}+\frac{F}{s+3}$$

$$s+1 = Cs(s+2)(s+3) + D(s+2)(s+3) + Es^2(s+3) + Fs^2(s+2)$$

$s=0$: $1 = D\cdot 2\cdot 3 \Rightarrow D = \frac{1}{6}$

$s=-2$: $-2+1 = E\cdot(-2)^2(-2+3) \Leftrightarrow -1 = 4E \Rightarrow E = -\frac{1}{4}$

$s=-3$: $-3+1 = F(-3)^2(-3+2) \Leftrightarrow -2 = -9F \Rightarrow F = \frac{2}{9}$

Equating the coefficient of s^3:

$$0 = C+E+F = C-\frac{1}{4}+\frac{2}{9} \Rightarrow$$

$$C = \frac{1}{4}-\frac{2}{9} = \frac{9-8}{36} = \frac{1}{36}$$

$\Downarrow$

$$\frac{s+1}{s^2(s+2)(s+3)} = \frac{1}{36s}+\frac{1}{6s^2}-\frac{1}{4(s+2)}+\frac{2}{9(s+3)}$$

$\Downarrow$

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2(s+2)(s+3)}\right\} = \frac{1}{36}+\frac{1}{6}t-\frac{1}{4}e^{-2t}+\frac{2}{9}e^{-3t} \Rightarrow$$ By the translation in t property

$$\mathcal{L}^{-1}\left\{e^{-s}\frac{s+1}{s^2(s+2)(s+3)}\right\} = u_1(t)\left(\frac{1}{36}+\frac{1}{6}(t-1)-\frac{1}{4}e^{-2(t-1)}+\frac{2}{9}e^{-3(t-1)}\right)$$

(iii) $\mathcal{L}^{-1}\left\{e^{-5s}\dfrac{4s+1}{s^2(s+2)(s+3)}\right\}$

First find $\mathcal{L}^{-1}\left\{\dfrac{4s+1}{s^2(s+2)(s+3)}\right\}$

$$\frac{4s+1}{s^2(s+2)(s+3)} = \frac{G}{s} + \frac{H}{s^2} + \frac{I}{s+2} + \frac{J}{s+3}$$

$$4s+1 = G\,s(s+2)(s+3) + H\,(s+2)(s+3) + I\,s^2(s+3) + J\,s^2(s+2)$$

$s=0$: $1 = H\cdot 2\cdot 3 \Rightarrow H = \frac{1}{6}$

$s=-2$: $-7 = I\cdot 4\cdot 1 \Rightarrow I = -\frac{7}{4}$

$s=-3$: $-11 = J\cdot 9\cdot(-1) \Rightarrow J = \frac{11}{9}$

Equating the coefficient of s^3:

$$0 = G + I + J = G - \frac{7}{4} + \frac{11}{9} \Rightarrow$$

$$G = \frac{7}{4} - \frac{11}{9} = \frac{63-44}{36} = \frac{19}{36}$$

$\Downarrow$

$$\frac{4s+1}{s^2(s+2)(s+3)} = \frac{19}{36s} + \frac{1}{6s^2} - \frac{7}{4(s+2)} + \frac{11}{9(s+3)}$$

$\Downarrow$

$$\mathcal{L}^{-1}\left\{\frac{4s+1}{s^2(s+2)(s+3)}\right\} = \frac{19}{36} + \frac{1}{6}t - \frac{7}{4}e^{-2t} + \frac{11}{9}e^{-3t} \Longrightarrow$$

by the translation in t property

$$\mathcal{L}^{-1}\left\{e^{-5s}\frac{4s+1}{s^2(s+2)(s+3)}\right\} = u_5(t)\left(\frac{19}{36} + \frac{1}{6}(t-5) - \frac{7}{4}e^{-2(t-5)} + \frac{11}{9}e^{-3(t-5)}\right)$$

$\Downarrow$ Combining (1), (2), (3) we get

$$\boxed{y(t) = 2e^{-2t} - 2e^{-3t} + u_1(t)\left(\frac{1}{36} + \frac{1}{6}(t-1) - \frac{1}{4}e^{-2(t-1)} + \frac{2}{9}e^{-3(t-1)}\right) - u_5(t)\left(\frac{19}{36} + \frac{1}{6}(t-5) - \frac{7}{4}e^{-2(t-5)} + \frac{11}{9}e^{-3(t-5)}\right)}$$

Problem 3 $\quad y'' + y = -\delta(t-\pi) + \delta(t-2\pi)\ ;\ y(0)=0,\ y'(0)=1$

Solution Apply the Laplace transform on both sides:

$$\mathcal{L}\{y\} = Y(s)$$
$$\mathcal{L}\{y''\} = s^2 Y(s) - s\,y(0) - y'(0) = s^2 Y(s) - 1$$
$$\mathcal{L}\{\delta(t-\pi)\} = e^{-\pi s}$$
$$\mathcal{L}\{\delta(t-2\pi)\} = e^{-2\pi s}$$

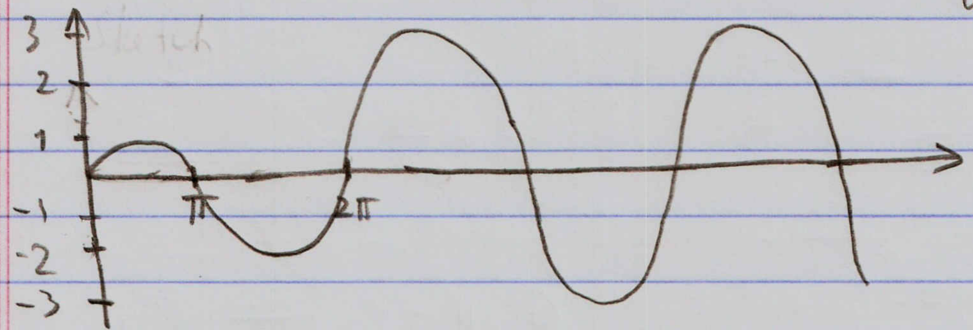
$\Downarrow$

$$(s^2+1)Y(s) - 1 = -e^{-\pi s} + e^{-2\pi s} \Rightarrow$$

$$Y(s) = \frac{1}{s^2+1} - \frac{e^{-\pi s}}{s^2+1} + \frac{e^{-2\pi s}}{s^2+1} \Rightarrow$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \sin t - u_\pi(t)\underbrace{\sin(t-\pi)}_{-\sin t} + u_{2\pi}(t)\sin(t-2\pi) =$$

$$= \sin t + u_\pi(t)\sin t + u_{2\pi}\sin t = \begin{cases} \sin t & 0 \le t < \pi \\ 2\sin t & \pi \le t < 2\pi \\ 3\sin t & t > 2\pi \end{cases}$$



Rem Note that $y(t) = 0$ for $t < 0$.

Problem 4 $\quad y'' + 6y' + 5y = e^t\delta(t-1)\ ;\ y(0)=0,\ y'(0)=4$

Solution $\quad$ Apply Laplace transform to both sides:

$$\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s)$$

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 4$$

$$\mathcal{L}\{e^t\delta(t-1)\} = \int_0^\infty e^t\delta(t-1)e^{-ts}\,dt = \int_{-\infty}^{\infty}\delta(t-1)\underbrace{e^te^{-ts}}_{f(t)}dt =$$

$$= f(1) = ee^{-s} (= e^{1-s})$$

$$\Downarrow$$

$$(s^2+6s+5)Y(s) - 4 = ee^{-s} \Rightarrow$$

$$Y(s) = \frac{4}{s^2+6s+5} + e\frac{e^{-s}}{s^2+6s+5}$$

$$\frac{1}{s^2+6s+5} = \frac{1}{(s+1)(s+5)} = \frac{1}{4}\left(\frac{1}{s+1} - \frac{1}{s+5}\right) \Rightarrow$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1} - \frac{1}{s+5} + \frac{e}{4}\ e^{-s}\left(\frac{1}{s+1} - \frac{1}{s+5}\right)\right\} =$$

$$= e^{-t} - e^{-5t} + \frac{e}{4}u_1(t)\left(e^{-(t-1)} - e^{-5(t-1)}\right) =$$

$$= \boxed{e^{-t} - e^{-5t} + \frac{e}{4}u_1(t)\left(e^{1-t} - e^{5-5t}\right)}$$

Problem 5 $\quad$ a) $\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\}$?

Let $\quad F(s) = \frac{1}{s^3}\ ,\ G(s) = \frac{1}{s^2+1}$

$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2}t^2 \qquad \mathcal{L}^{-1}(G(s)) = \sin t \Rightarrow$ by Convolution Theorem

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \frac{1}{2}t^2 * \sin t = \frac{1}{2}\int_0^t (t-\tau)^2 \sin\tau\, d\tau =$$

$$= \frac{1}{2}\int_0^t \left(t^2\sin\tau - 2t\tau\sin\tau + \tau^2\sin\tau\right)d\tau = (*)$$

$$\int_0^t \sin\tau\, d\tau = -\cos\tau \Big|_0^t = 1-\cos t \qquad (1)$$

$$\underbrace{\int_0^t \tau\sin\tau\, d\tau = -\tau\cos\tau \Big|_0^t + \int_0^t \cos\tau\, d\tau}_{\text{Integration by parts}} = -t\cos t + \sin\tau \Big|_0^t =$$

$$= -t\cos t + \sin t$$

$$\int_0^t \tau^2 \sin\tau\, d\tau = -\tau^2\cos\tau\Big|_0^t + 2\int_0^t \tau\cos\tau\, d\tau = -t^2\cos t + 2\tau\sin\tau\Big|_0^t -$$

$$-2\int_0^t \sin\tau\, d\tau = -t^2\cos t + 2t\sin t - 2(1-\cos t) =$$

$$= -t^2\cos t + 2t\sin t - 2 + 2\cos t$$

$$(*) = \frac{1}{2}\left[t^2(1-\cos t) - 2t(-t\cos t + \sin t) - t^2\cos t + 2t\sin t - 2 + 2\cos t\right] =$$

$$= \frac{1}{2}\left[t^2 - \cancel{t^2\cos t} + \cancel{2t^2\cos t} - \cancel{2t\sin t} - \cancel{t^2\cos t} + \cancel{2t\sin t} - 2 + 2\cos t\right] =$$

$$= \frac{1}{2}\left(t^2 - 2 + 2\cos t\right) = \boxed{\frac{1}{2}t^2 - 1 + \cos t}$$

b) $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$

Let $F(s) = \frac{s}{s^2+1}$, $G(s) = \frac{1}{s^2+1} \Rightarrow$

$\mathcal{L}^{-1}\{F(s)\} = \cos t$, $\mathcal{L}^{-1}\{G(s)\} = \sin t \Rightarrow$ By convolution theorem

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \cos t * \sin t = \int_0^t \cos(t-\tau)\sin\tau\, d\tau$$

From trigonometry

$\sin\alpha \cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right) \Rightarrow$

$\sin \underbrace{\tau}_{\alpha} \cos \underbrace{(t-\tau)}_{\beta} = \frac{1}{2}\left(\sin(\cancel{\tau}+t-\cancel{\tau}) + \sin(\tau-(t-\tau))\right) =$

$= \frac{1}{2}\left(\sin t - \sin(2\tau - t)\right) \Rightarrow$

$\int_0^t \cos(t-\tau)\sin\tau \, d\tau = \frac{1}{2}\int_0^t \left(\sin t - \sin(2\tau-t)\right) dt =$

$= \frac{1}{2} t \sin t + \frac{1}{2} \left.\frac{\cos(2\tau - t)}{2}\right|_0^t = \frac{1}{2} t \sin t + \frac{1}{4}\underbrace{\left(\cos t - \cos(-t)\right)}_{0}$

$= \boxed{\frac{1}{2} t \sin t}$

Problem 6

6 a) $y'' - 2y' + 5y = g(t); \quad y(0) = 0, y'(0) = 2$

Apply Laplace transform to both sides

$\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s)$

$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2$

Let $G(s) = \mathcal{L}\{g(t)\}$

Then $(s^2 - 2s + 5)Y(s) - 2 = G(s) \Rightarrow$

$$Y(s) = \frac{G(s)}{s^2-2s+5} + \frac{2}{s^2-2s+5}$$

$s^2 - 2s + 5 = s^2 - 2s + 1 + 4 = (s-1)^2 + 4 \Rightarrow$

$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+4}\right\} = \frac{1}{2}e^t \sin 2t \Rightarrow$ By convolution theorem

$y(t) = \frac{1}{2}e^t \sin 2t * g(t) + e^t \sin 2t =$

$$= \boxed{\frac{1}{2}\int_0^t e^{t-\tau}\sin 2(t-\tau)\, g(\tau)\, d\tau + e^t \sin 2t}$$

6b) From the method of variation of parameter

$$y(t) = y_1(t)\int_0^t \frac{-g(\tau)y_2(\tau)}{W(y_1,y_2)(\tau)}\, d\tau + y_2(t)\int_0^t \frac{g(\tau)\, y_1(\tau)}{W(y_1,y_2)(\tau)}\, d\tau + C_1 y_1(t) + C_2 y_2(t)$$

where $y_1(t)$ and $y_2(t)$ is a fundamental set of solutions of the homogeneous equation

$$y'' - 2y' + 5y = 0$$ and C_1 and C_2 are some constants

The characteristic equation is

$$r^2 - 2r + 5 = 0 \Leftrightarrow (r-1)^2 + 4 = 0 \Rightarrow$$

$$r_{1,2} = 1 \pm 2i$$

$$\left(\text{also } D = 4 - 20 = -16 \Rightarrow r_{1,2} = \frac{2 \pm 4i}{2} = 1 \pm 2i\right)$$

We can take

$$y_1(t) = e^t \cos 2t, \quad y_2(t) = e^t \sin 2t$$

$$W = \begin{vmatrix} e^t\cos 2t & e^t \sin 2t \\ -2e^t \sin 2t + e^t \cos 2t & 2e^t \cos 2t + e^t \sin 2t \end{vmatrix} =$$

$$= \begin{vmatrix} e^t\cos 2t & e^t \sin 2t \\ -2e^t \sin 2t & 2e^t \cos 2t \end{vmatrix} = 2e^{2t}(\cos^2 2t + \sin^2 2t) = 2e^{2t}$$

Substituting into the formula $(**)$ we get

$$y(t) = e^t \cos 2t \int_0^t \frac{-e^\tau \sin 2\tau\, g(\tau)}{2e^{2\tau}}\, d\tau + e^t \sin 2t \int_0^t \frac{e^\tau \cos 2\tau\, g(\tau)}{2e^{2\tau}}\, d\tau + C_1 e^t \cos 2t + C_2 e^t \sin 2t$$

$$= \frac{1}{2}\int_0^t e^{t+\tau-2\tau}\Big(\underbrace{\sin 2t \cos 2\tau - \cos 2\tau \sin 2\tau}_{\sin 2(t-\tau)}\Big) g(\tau)\,d\tau +$$

$$+ C_1 e^t \cos 2t + C_2 e^t \sin 2t = \frac{1}{2}\int_0^t e^{t-\tau}\sin 2(t-\tau)\, g(\tau)\,d\tau +$$

$$+ C_1 e^t \cos 2t + C_2 e^t \sin 2t$$

Now find C_1 and C_2 from the initial conditions

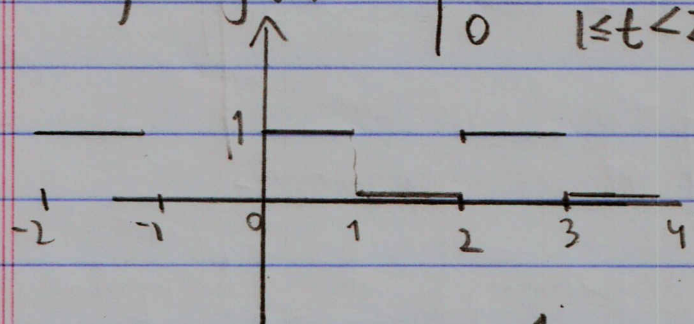
$y(0) = 0 \Rightarrow \quad C_1 = 0$

$y'(0) = 2 \Rightarrow \quad 2C_2 = 2 \Rightarrow C_2 = 1$ (note that $\frac{1}{2}\int_0^t e^{t-\tau}\sin 2(t-\tau) g(\tau)\,d\tau$ is exactly the solution of our non hom. equation with zero initial conditions) $\Rightarrow$

$$\boxed{y(t) = \frac{1}{2}\int_0^t e^{t-\tau}\sin 2(t-\tau)\, g(\tau)\,d\tau + e^t \sin 2t}$$

exactly as in item a)

Problem 7 a) $f(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & 1 \le t < 2 \end{cases}$ and $f(t)$ has period 2



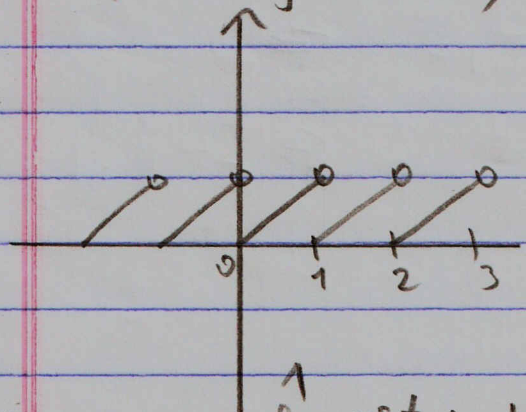
$T = 2$

$$\int_0^2 e^{-st} f(t)\,dt = \int_0^1 e^{-st}\,dt = \frac{e^{-st}}{-s}\Big|_0^1 = \frac{1-e^{-s}}{s} \Rightarrow$$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} f(t)\,dt = \frac{1-e^{-s}}{(1-e^{-2s})\,s} =$$

$$= \boxed{\frac{1}{(1+e^{-s})\,s}}$$

Problem 7 b) $f(t) = t$, $0 \le t < 1$ and $f(t)$ has period 1



$T = 1$: $\int_0^1 e^{-st}\, t\, dt \underset{\text{by parts}}{=} \frac{te^{-st}}{-s}\Big|_0^1 + \int_0^1 \frac{e^{-st}}{s}\, dt =$

$$= -\frac{e^{-s}}{s} - \frac{e^{-st}}{s^2}\Big|_0^1 = -\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} = \frac{1-e^{-s}}{s^2} - \frac{e^{-s}}{s} = \frac{1-se^{-s}-e^{-s}}{s^2}$$

$$\mathcal{L}\{f(t)\} = \boxed{\frac{1}{1-e^{-s}} \cdot \frac{1-se^{-s}-e^{-s}}{s^2}}$$