Pushing the Limits

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What is a Limit?

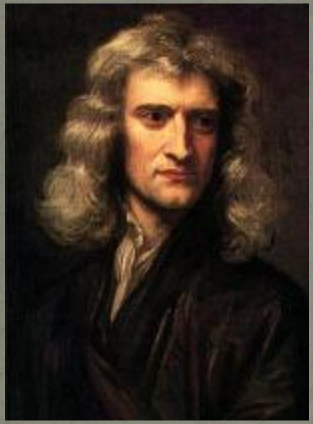
- A speed limit is the maximum speed you can drive without breaking the law.
- The mathematical definition of a limit is "The value of f(x), as x approaches A, equals L."
- Limits are the foundation of calculus, from which derivatives, integrals, velocities, and accelerations can be calculated.

History

"Archimedes of Syracuse first developed the idea of limits to measure curved figures and the volume of a sphere in the third century B.C. By carving these figures into small pieces that can be approximated, then increasing the number of pieces, the limit of the sum of pieces can give the desired quantity. Archimedes' thesis, The Method, was lost until 1906, when mathematicians discovered that Archimedes came close to discovering infinitesimal calculus. As Archimedes' work was unknown until the twentieth century, others developed the modern mathematical concept of limits. Englishman Sir Isaac Newton and German Gottfried Wilhelm von Leibniz independently developed the general principles of calculus (of which the theory of limits is an important part) in the seventeenth century."

http://science.jrank.org/pages/3934/Limit-History.html **

Newton vs. Leibniz



Isaac Newton first discovered Calculus in England in 1666, but wrote little on it until decades later. Meanwhile in Germany, Gottfriend Leibniz began working on his variant form of calculus in 1674 and published it in 1684.

There was great controversy and bitterness about who discovered calculus first, but eventually history sided with Newton. While Leibniz was not the first to discover it, he contributed important concepts to calculus.

Artillery Round

Suppose that an artillery round is fired from a gun and its position r(t) at time t is

$$\vec{r}(t) = (321t)i + (250t - 4.9t^2)j$$

Where distance is measured in feet and time in seconds. Find the instantaneous velocity of the shell after 40 seconds.

Artillery Round Solution:

The instantaneous velocity (rate of change of position with respect to time) is the limit of the average velocities over successively smaller periods of time. Here the average velocity between times t=40 and t=40+h is given by

average velocity = change in position/time elapsed

$$= \frac{r(40+h)-r(40)}{h} \qquad = \frac{1}{h}[r(40+h)-r(40)]$$

Artillery Round

Time Interval (sec)	h (sec)	$=\frac{1}{h}[r(40+h)-r(40)]$
40.1	.1	321*i-142.49*j
40.01	.01	321*i-142.049*j
40.001	.001	321*i-142.0049*j
40.0001	.0001	321*i-142.00049*j

Artillery Round

Using this table, one can see that as the value of h decreases, the result is closer and closer to the right answer. This use of limits allows one to see from a wider perspective the behavior of the function as it approaches the limit.

So the final answer using the table is instantaneous velocity at time t = 40 *seconds* is

v = 321*i – 142*j meters/second

Methods to Solve Limits

- Direct Substitution
- Factoring
- Rationalizing

Direct Substitution

$$\lim_{x \to 5} \frac{x-7}{x+2}$$

= (5+7)/(5+2)

= 12/7

If x is in the domain of the function f(x), to find the limit we can simply substitute the value of a in for the value of x.

For vector functions like the Artillery Round Example, one can use direct substitution

Factoring

- In some cases x is not in the domain of f(x) and direct substitution is not an option.

$$\lim_{x\to 8} \frac{x^2 + 12x + 32}{x^2 + 6x - 16}$$

As you can see, if we substitute the value of a in for x, the function will not exist.

In this case we first factor.

$$\frac{x^2 + 12x + 32}{x^2 + 6x - 16} = \frac{(x+8)(x+4)}{(x+8)(x-2)}$$

the (x+8)'s will cancel out giving us

$$\frac{x+4}{x-2}$$

so

$$\lim_{x\to 8} \frac{x+4}{x-2} = \frac{8+4}{8-2} = \frac{12}{6} = 2$$

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Rationalizing

- In other cases factoring is not immediately available and the form of the equation must first be changed.

$$\lim_{x\to 1}\frac{\sqrt{x^2+x+23}-5}{x-1}$$

As you can see, if we try to substitute x for 1, the bottom goes to zero and the function is no longer defined. Also we cannot factor the part of the function under the square root easily so we must change its form.

We do this by multiplying by the reciprocal of the numerator in order to eliminate the square root.

$$\lim_{x\to 1}\frac{\sqrt{x^2+x+23}-5}{x-1}\bullet\frac{\sqrt{x^2+x+23}+5}{\sqrt{x^2+x+23}+5}$$

We can do this because the reciprocal over itself is 1, and we are not changing the value of the function.

After this step we have

$$\lim_{x\to 1}\frac{x^2+x+23-25}{(x-1)(\sqrt{x^2+x+23}+5)}$$

the numerator can be rewritten as

$$x^2+x-2=(x-1)(x+2)$$

the function is now

$$\lim_{x\to 1}\frac{(x-1)(x+2)}{(x-1)(\sqrt{x^2+x+23}+5)}$$

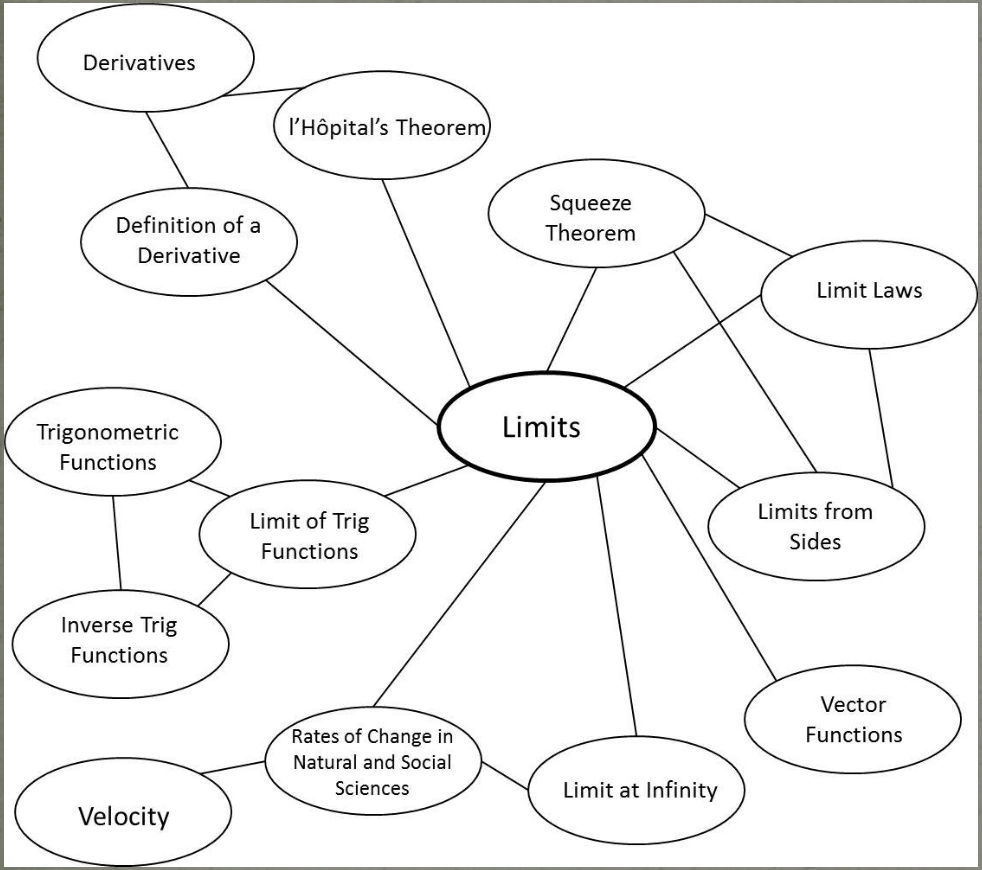
the (x-1)'s cancel leaving us with

$$\lim_{x\to 1}\frac{x+2}{\sqrt{x^2+x+23}+5}$$

now we can plug in the value of a

$$\frac{1+2}{\sqrt{1^2+1+23}+5}=\frac{3}{\sqrt{25}+5}=\frac{3}{10}$$

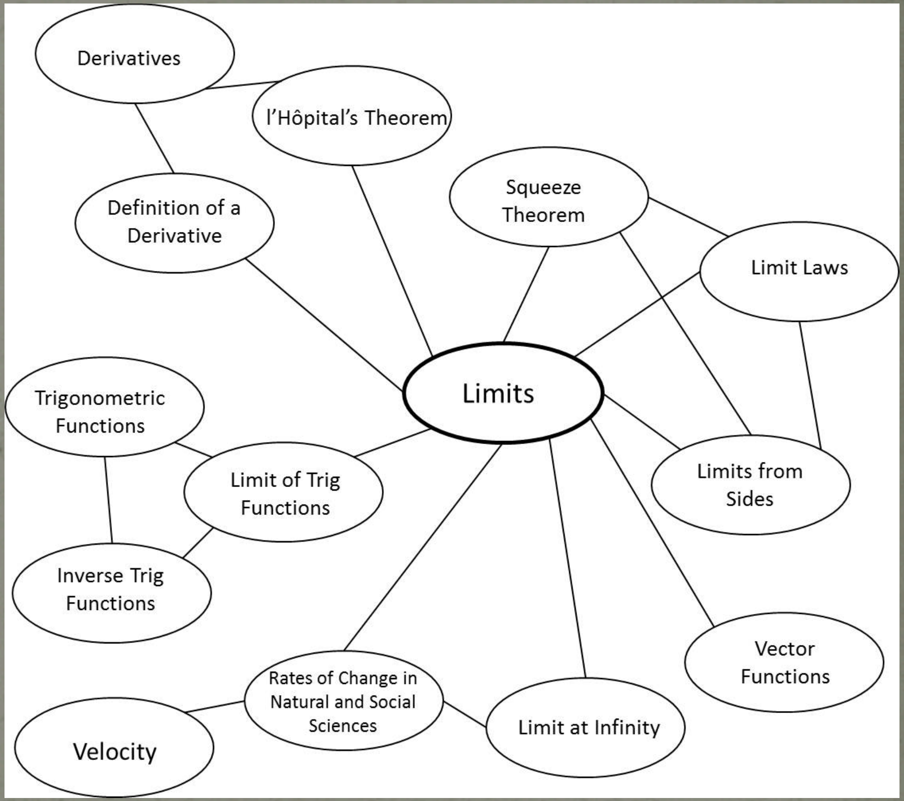
Visual Map



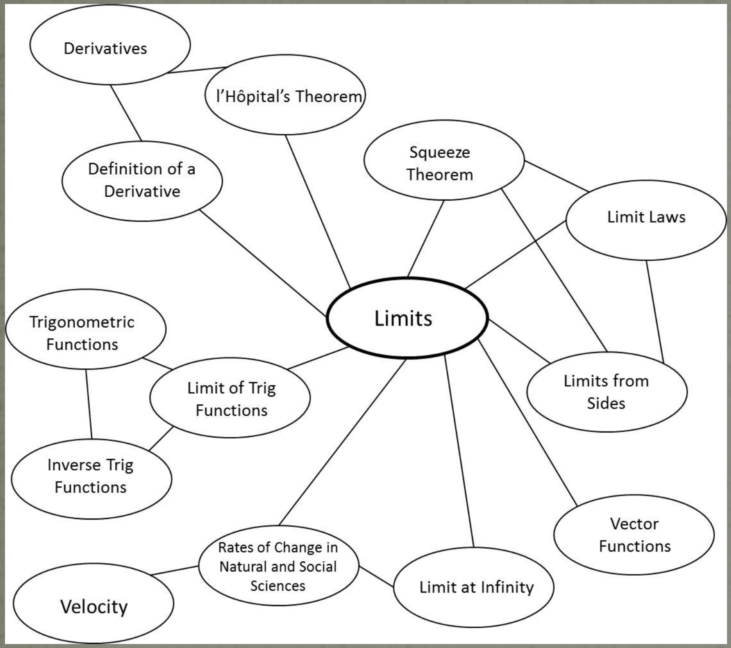
- In the center we have the "Limits", which is the base for this visual map. Branching out from it are the sub-topics that are most important to the concept of limits.
- Certain sub-topics are connected not only with the base "Limits", but also with themselves.

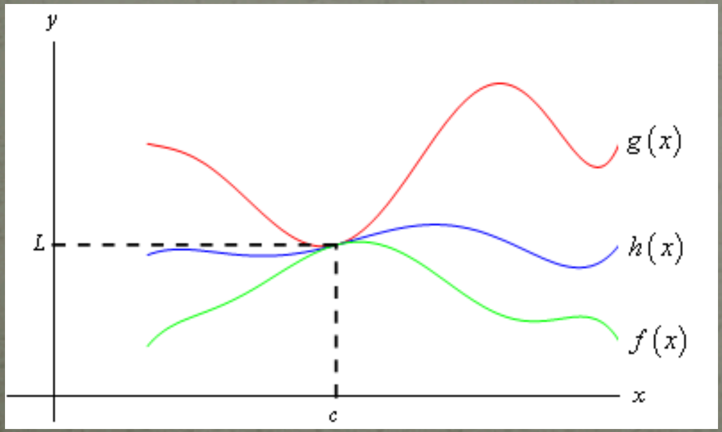
Visual Map

- Limits at infinity and rates of change in the natural/social sciences are connected because many things approach a specific number when the limit of their function goes to infinity such as:
  - Population growth
  - Temperature,
  - Terminal velocity
- Topics such as the definition of a derivative and limits of trig functions have other topics connected to them that are not directly related to limits.



Visual Map





- For example, the concept of limits is connected to the definition of derivative, which is a limit defined by

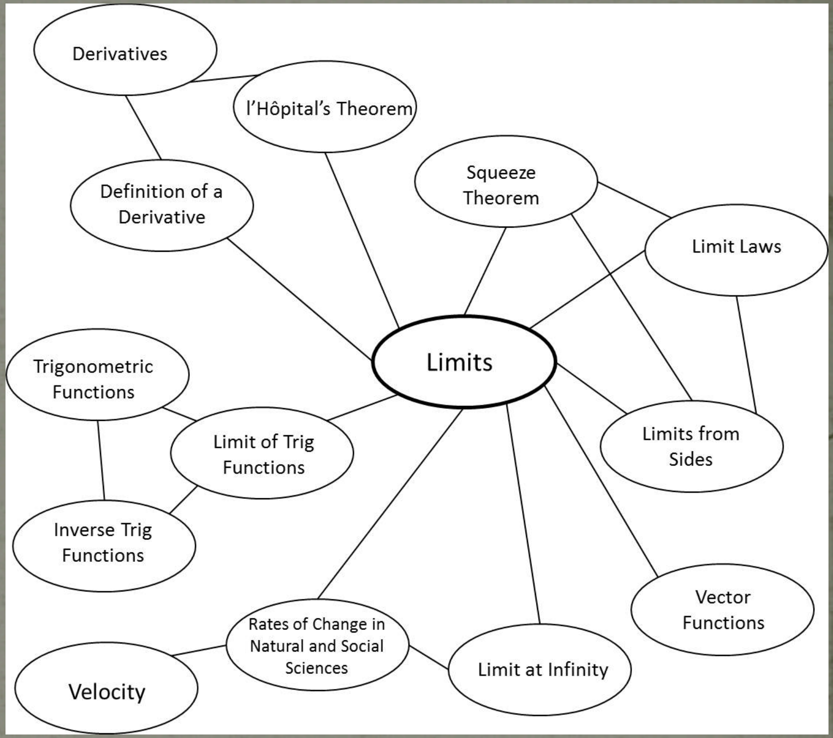
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- Squeeze theorem is connected to limits from sides because if you have 2 functions whose limits as they approach a point a are the same, and if there is another function that is in between the two, they will "squeeze" it in from both sides.

Visual Map

- This forces the limit of the third function as it approaches point a to be the same as the other 2
- For example, the limits section is connected to the definition of derivative, which is a limit defined by

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



- Overall, the concepts of limits are all interconnected in one way or another, and they come together to form a vital part of the calculus, mathematics, and the real world.

Examples

Skydiver:

Skydiver

- A recent breakthrough in the field of anti-gravity devices has resulted in a device that can reduce the effects of gravity on an object. Scientists have decided to test the effectiveness of this device by having a skydiver jump from a plane with the device and track his velocity as he descends. His velocity is given by

$$\frac{t\sqrt{3}-\sqrt{29t+84}}{t-12}$$

- At what velocity is the skydiver falling at time t =12? Does the device appear to work?

At t=12 the function is indeterminate so we must instead find the limit t->12 of f(x)

$$\lim_{t\to 12}\frac{t\sqrt{3}-\sqrt{29t+84}}{t-12}$$

Skydiver

- First we will multiply by the conjugate in order to eliminate the radical

$$\lim_{t\to 12}\frac{t\sqrt{3}-\sqrt{29t+84}}{t-12}*\frac{t\sqrt{3}+\sqrt{29t+84}}{t\sqrt{3}+\sqrt{29t+84}}$$

$$\lim_{t\to 12}\frac{(t\sqrt{3})^2-\sqrt{29t+84}^{\,2}}{(t-12)(t\sqrt{3}+\sqrt{5t+84})} = \lim_{t\to 12}\frac{3t^2-29t-84}{(t-12)(t\sqrt{3}+\sqrt{29t+84})}$$

- Now we can factor the numerator and cancel out the common factors which gives us

$$\lim_{t\to 12}\frac{(t-12)(3t+7)}{(t-12)(t\sqrt{3}+\sqrt{29t+84})} = \lim_{t\to 12}\frac{3t+7}{(t\sqrt{3}+\sqrt{29t+84})}$$

Skydiver

- Now that we have eliminated the (t-12)'s, we can simply plug in 12 for t:

$$\frac{(3*12)+7}{(12\sqrt{3}+\sqrt{5(12)+84})} = \frac{36+7}{\sqrt{432}+\sqrt{432}}$$

$$\frac{36+7}{\sqrt{432}+\sqrt{432}} = \frac{43}{41.57} = 1.03\frac{m}{s}$$

Skydiver

- In order to see if this answer makes sense we will calculate the velocity of a skydiver when t=12 with the normal effects of gravity. This velocity is given by 9.8t.

$$v(t) = 9.8t$$

- If we Plug in 12 for t we have:

$$9.8(12) = 117.6\frac{m}{s}$$

Skydiver

- Because the first velocity is a mere fraction of the second, only .87%, we can safely conclude that yes, the anti-gravity device appears to work very effectively

Shark Population

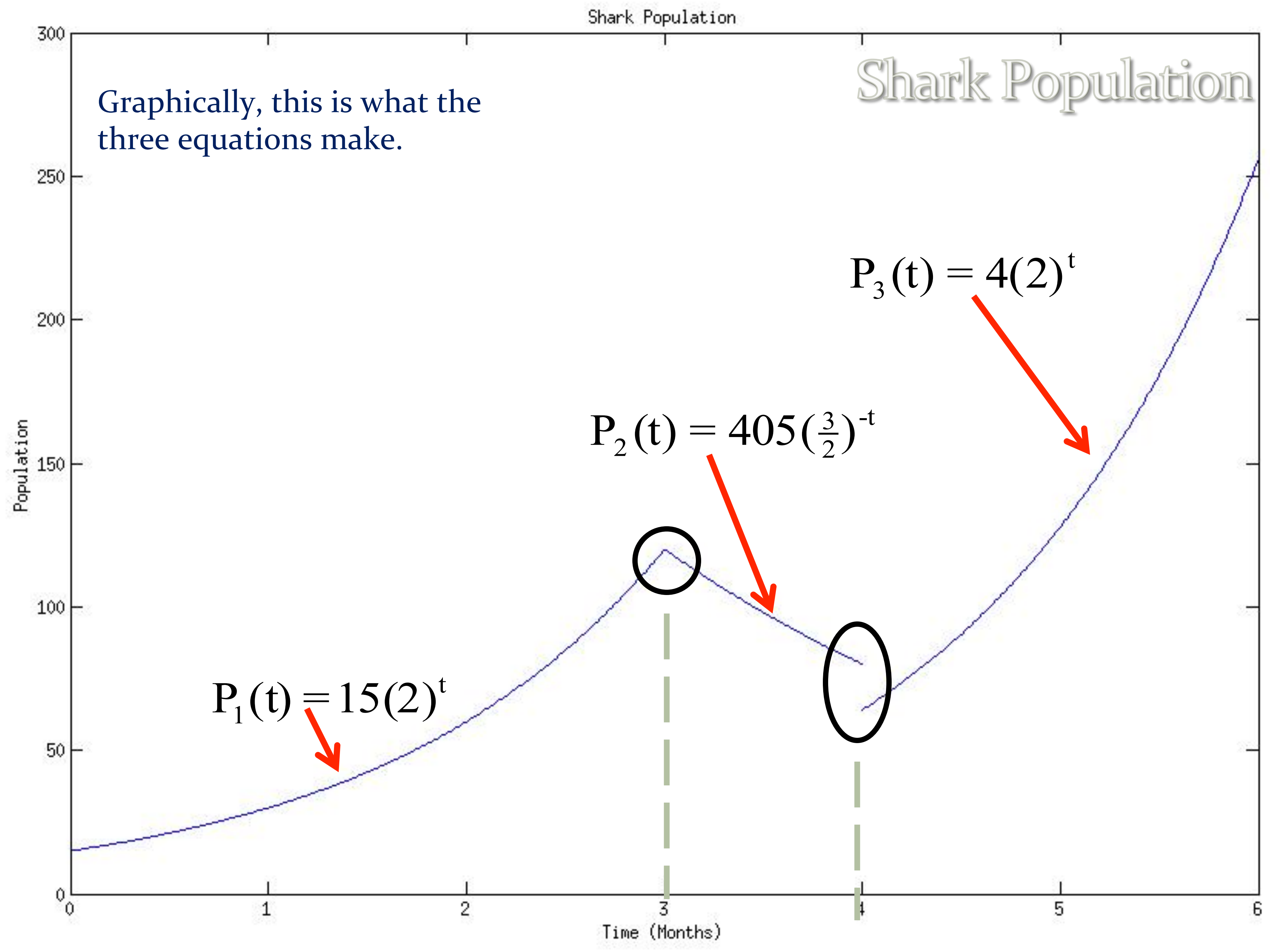
- Fifteen sharks were introduced into a saltwater lake, where they grew exponentially for three months
- After 3 months, some of the shark population obtained a fatal infection. After one more month, human scientists took notice and removed 16 infected sharks from the lake, allowing the shark's numbers to return to a normal growth pattern
- The following function represents the population, where t is in months:

$$F(t) = \begin{cases} P_1 = 15(2)^t, [0,3] \\ P_2 = 405(\tfrac{3}{2})^{-t}, [3,4) \\ P_3 = 4(2)^t, [4,6] \end{cases}$$

What is the population at 3 months? At 4 months?

Shark Population

Graphically, this is what the three equations make.



Shark Population

What is the Population after 3 months?

The first thing we have to figure out is what exactly is being asked. The population after three months is the same as the limit after 3 months, so we are really being asked this:

What is the limit of f(t) as t = 3?

Since the point at **t = 3** *is where two different equations are defined, we need to plug in* **t = 3** *for both to check if the limit exists. In other words, we simply use direct substitution for both P1 and P2 and see if they are equal.*

$$\lim_{t \to 3} f(t) = ?$$

$$P_1(t) = \lim_{t \to 3} 15(2)^t$$
$$P_1(3) = 15(2)^3$$
$$P_1(3) = 15 \bullet 8$$
$$P_1(3) = 120$$

$$P_2(t) = \lim_{t \to 3} 405\left(\tfrac{3}{2}\right)^{-t}$$
$$P_2(3) = 405\left(\tfrac{3}{2}\right)^{-3}$$
$$P_2(3) = 405 \bullet \left(\tfrac{8}{27}\right)$$
$$P_2(3) = 120$$

Because both equations are equal to 120 at **t = 3*****, the function is continuous and we can say that*** **f(3) = 120.**

$$\lim_{t \to 3} f(t) = 120$$

Shark Population

What is the population after 4 months?

This is similar to last question, but this is after four months. So what it is really asking us is:

What is the limit of f(t) at t = 4?

We again need to check if f(t) is continuous at t = 4, if both values are equal in P2 and P3, then it is continuous and the limit is defined.

$$\lim_{t \to 4} f(t) = ?$$

$$P_2(t) = \lim_{t \to 4} 405\left(\tfrac{3}{2}\right)^{-t}$$
$$P_2(4) = 405\left(\tfrac{3}{2}\right)^{-4}$$
$$P_2(3) = 405 \bullet \left(\tfrac{16}{81}\right)$$
$$P_2(4) = 120$$

$$P_3(t) = \lim_{t \to 4} 4(2)^{t}$$
$$P_3(4) = 4(2)^{3}$$
$$P_3(4) = 4 \bullet (16)$$
$$P_3(4) = 64$$

Since the values are different, the function does not have a limit at t = 4, but it does have limits from the sides. So mathematically,

$$\lim_{t \to 4} f(t) = Undef.$$

But,

$$\lim_{t \to 4-} f(t) = 80$$

$$\lim_{t \to 4+} f(t) = 64$$

Youtube Video Views

- The popular video website Youtube tracks the number of views per week of each of its videos. A recent user posted a video of a dancing monkey. This video became very popular, and the number of views per week rose. The number of views per week of this particular video is modeled by the function:

$$V(x) = \frac{25x^2}{2e^{.1x}}$$

- After a long time, (t=infinity) how many views per week will this video have?

Youtube Video Views

- Solution: We set up our limit equation as follows:

$$\lim_{x\to\infty}\frac{100x^2}{8e^{.1x}}$$

- If we try direct substitution, we get $\frac{\infty}{\infty}$, and since that is an indeterminate form, we must use another technique.

Youtube Video Views

- To solve this limit, we will use L'Hôpital's rule, which says that

$$\lim_{x \to n} \frac{f(x)}{g(x)}$$

- is equal to the derivative of f(x) over the derivative of g(x), or

$$\lim_{x \to n} \frac{f'(x)}{g'(x)}$$

Youtube Video Views

- Using derivatives, we will say that:

$$\lim_{x\to\infty}\frac{\frac{d}{dx}(100x^2)}{\frac{d}{dx}(8e^{.1x})}=\lim_{x\to\infty}\frac{200x}{.8e^{.1x}}$$

- Here, if we try plugging in ∞ to the equation, we still get ∞ over ∞, so we will try using l'Hôpital's rule again.

Youtube Video Views

- l'Hôpital's (the second time)

$$\lim_{x\to\infty}\frac{\frac{d}{dx}(200x)}{\frac{d}{dx}(.8e^{.1x})}=\lim_{x\to\infty}\frac{200}{.08e^{.1x}}$$

- Now we have another function. If we try plugging in ∞ one more time, we end up with 200/∞, which we can call zero.

$$\frac{200}{\infty}=0$$

Youtube Video Views

- But what does this all mean? It means that after a long time, this video loses its popularity, and goes from around 670 views per week at its highest amount, all the way down to zero views per week.

References

http://science.jrank.org/pages/3934/Limit-History.html

http://www.sosmath.com/diffeq/first/application/newton/newton.html

http://www.khanacademy.org/exercisedashboard