REVIEW: Power Series

DEFINITION 1. *A power series about $x = x_0$ (or centered at $x = x_0$), or just* **power series**, *is any series that can be written in the form*

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

where x_0 and a_n are numbers. The a_n's are called the **coefficients** *of the power series.*

Absolute Convergence: The series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is said to converge absolutely at x if $\sum_{n=0}^{\infty} |a_n|\,|x - x_0|^n$ converges.

If a series converges absolutely then it converges (But in general not vice versa).

EXAMPLE 2. *The series $\sum_{n=1}^{\infty} \frac{x^n}{n} = \lim_{m\to\infty}$ converges at $x = -1$, but it doesn't converges absolutely:*

$$1 - \frac{1}{2} + \frac{1}{3} - \ldots = \ln 2$$

but

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots$$

is divergent.

Fact: If the series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ converges absolutely at $x = x_1$ then it converges absolutely for all x such that $|x - x_0| < |x_1 - x_0|$

THEOREM 3. *For a given power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ there are only 3 possibilities:*

1. *The series converges only for $x = x_0$.*
2. *The series converges for all x.*
3. *There is $R > 0$ such that the series converges if $|x - x_0| < R$ and diverges if $|x - x_0| > R$. We call such R the* **radius of convergence.**

REMARK 4. In case 1 of the theorem we say that $R = 0$ and in case 2 we say that $R = \infty$

How to find Radius of convergence: If $a_n \neq 0$ for any n and $\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$ exists, then

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

Geometric Series: $1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n = \lim_{m\to\infty} \sum_{n=1}^{m} x^n = \lim_{m\to\infty} \sum_{n=1}^{m} \frac{1-x^{m+1}}{1-x} = \frac{1}{1-x}$ provided $|x| < 1$. The series diverges if $|x| \geq 1$. We can use also the ratio test: $a_n = 1$ and then $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = 1$

Another example: for $\sum_{n=1}^{\infty} \frac{x^n}{n}$ we have $a_n = \frac{1}{n}$ and thus

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} = \lim_{n\to\infty} \frac{n+1}{n} = 1$$

The Taylor series for $f(x)$ about $x = x_0$

Assume that f has derivatives of any order at $x = x_0$. Then

$$f(x) = \sum_{n=0}^{m} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(m+1)}(c)}{(m+1)!}(x - x_0)^{m+1}$$

where c is between x and x_0. The remainder converges to zero at least as fast as $(x - x_0)^{m+1}$ when $x \to x_0$. Formally we can consider the following power series:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$