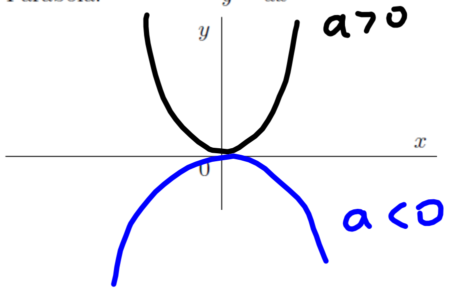
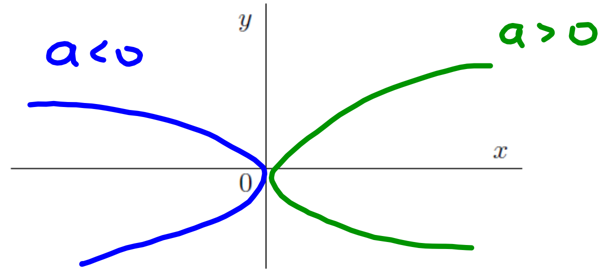
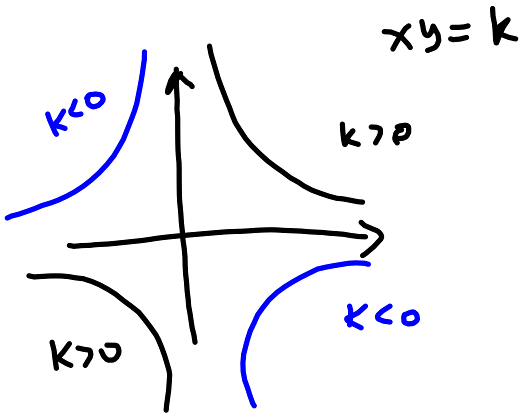
11.5: Quadric surfaces

REVIEW: Parabola, hyperbola and ellipse.

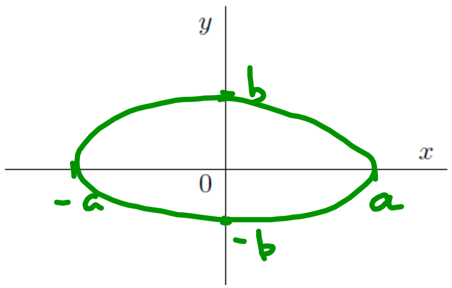
- Parabola: $y = ax^2$ or $x = ay^2$.





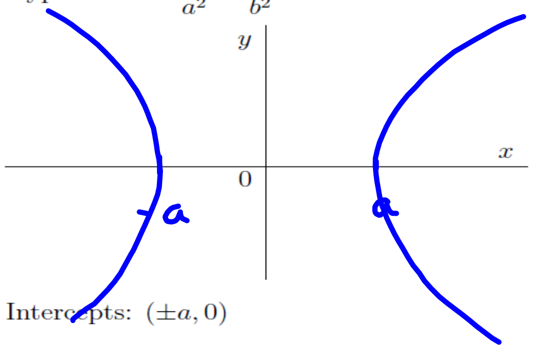


- Ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

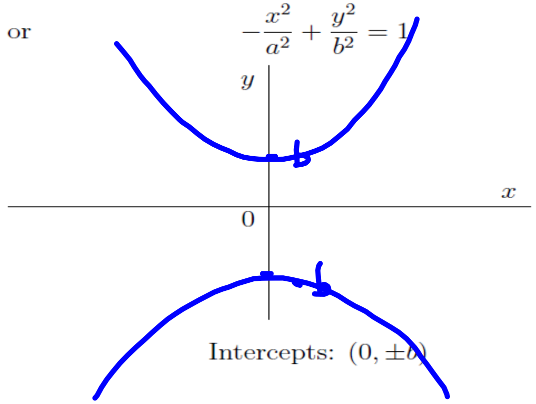


Intercepts: $(\pm a, 0) \& (0, \pm b)$

- Hyperbola: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ or $-\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$



Intercepts: $(\pm a, 0)$



Intercepts: $(0, \pm b)$

The most general second-degree equation in three variables x, y and z:

$$Ax^2 + By^2 + Cz^2 + axy + bxz + cyz + d_1x + d_2y + d_3z + E = 0, \tag{1}$$

where $A, B, C, a, b, c, d_1, d_2, d_3, E$ are constants. The graph of (1) is a quadric surface.

Note if $A = B = C = a = b = c = 0$ then (1) is a linear equation and its graph is a plane (this is the case of degenerated quadric surface).

By translations and rotations (1) can be brought into one of the two standard forms:

$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0.$$

In order to sketch the graph of a surface determine the curves of intersection of the surface with planes parallel to the coordinate planes. The obtained in this way curves are called **traces** or **cross-sections** of the surface.

Quadric surfaces can be classified into 5 categories:

ellipsoids, hyperboloids, cones, paraboloids, quadric cylinders. (shown in the table, see Appendix.)

The elements which characterize each of these categories:

1. Standard equation.
2. Traces (horizontal (by planes $z = k$), yz-traces (by $x = 0$) and xz-traces (by $y = 0$).
3. Intercepts (in some cases).

To find the equation of a trace substitute the equation of the plane into the equation of the surface (cf. Example 4, Section 1.1 notes). Note, in the examples

$a > 0, \; b > 0, \; c > 0$

below *the constants* $a, b,$ *and* c *are assumed to be positive.*

- Ellipsoid. Standard equation:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Note if $a=b=c$ we have a sphere.

EXAMPLE 1. *Sketch the ellipsoid*

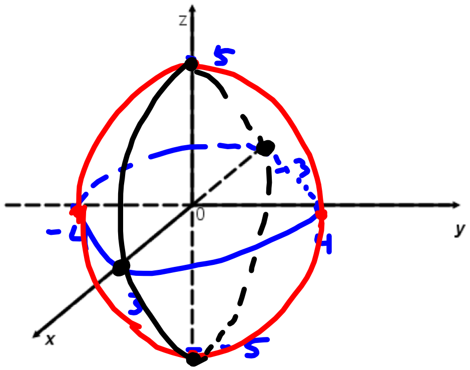
$$\frac{x^2}{9}+\frac{y^2}{16}+\frac{z^2}{25}=1$$

Solution

– Find intercepts:
  * x-intercepts: if $y=z=0$ then $x=\pm 3$ $(\pm 3, 0, 0)$
  * y-intercepts: if $x=z=0$ then $y=\pm 4$ $(0, \pm 4, 0)$
  * z-intercepts: if $x=y=0$ then $z=\pm 5$ $(0, 0, \pm 5)$

– Obtain traces of:
  * the xy-plane: plug in $z=0$ and get $\frac{x^2}{9}+\frac{y^2}{16}=1$ ellipse
  * the yz-plane: plug in $x=0$ and get $\frac{y^2}{16}+\frac{z^2}{25}=1$
  * the xz-plane: plug in $y=0$ and get

$$\frac{x^2}{9}+\frac{z^2}{25}=1$$



The solution is posted in key for Quiz 3!

- Hyperboloids: There are two types:
  - *Hyperboloid of one sheet.*

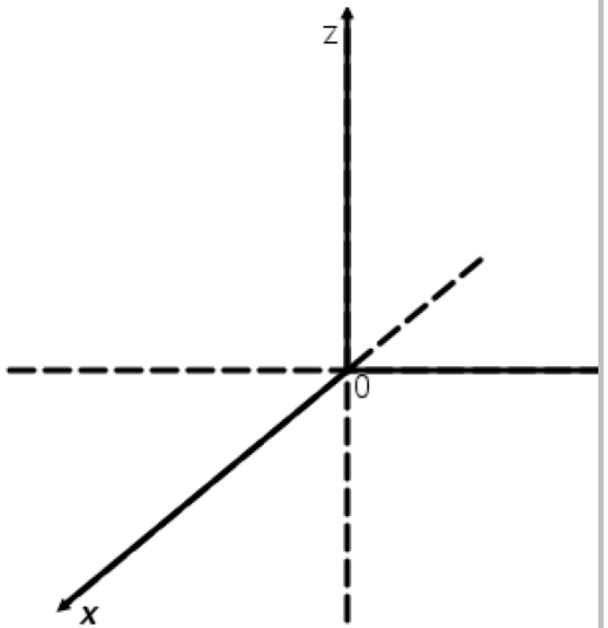
    Standard equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

EXAMPLE 2. *Sketch the hyperboloid of one sheet*

$$x^2 + y^2 - \frac{z^2}{9} = 1$$

Plane	Trace
$z = 0$	
$z = \pm 3$	
$x = 0$	
$y = 0$	



– *Hyperboloid of two sheets.* **The solution is posted in key for Quiz 3!**

Standard equation:

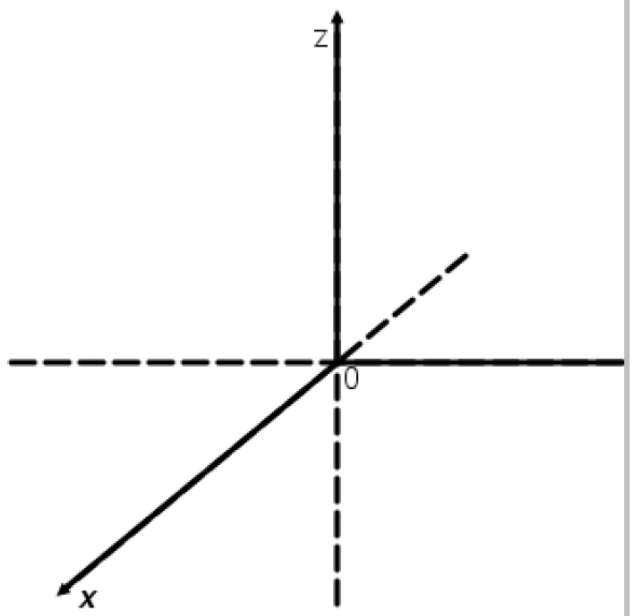
$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

EXAMPLE 3. *Sketch the hyperboloid of two sheet*

$$-x^2-\frac{y^2}{9}+z^2=1$$

Solution Find z-intercepts: if $x=y=0$ then $z=$

Plane	Trace
$z=\pm 2$	
$x=0$	
$y=0$	



- Elliptic Cones. Standard equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

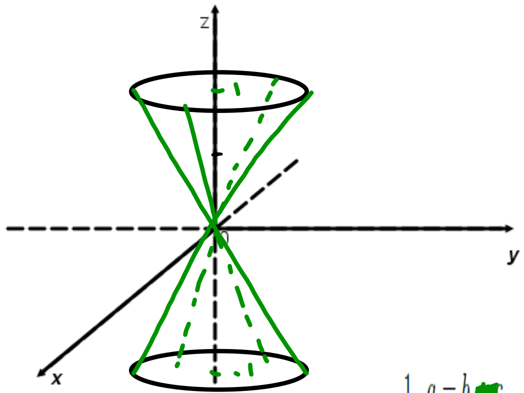
If $a = b = c$ then we say that we have a *circular cone.*

EXAMPLE 4. *Sketch the elliptic cone*

$$z^2 = x^2 + \frac{y^2}{9}$$

Plane	Trace
$z = \pm 1$	$x^2 + \frac{y^2}{9} = 1$ ellipse
$x = 0$	$z^2 = \frac{y^2}{9} \Rightarrow z = \pm \frac{y}{3}$ two lines
$y = 0$	$z^2 = x^2 \Rightarrow z = \pm x$ two lines

$(\pm 1)^2 = 1$



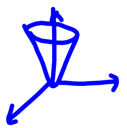
Special cases:

1. $a = b$ circular cone

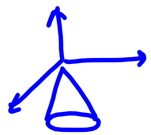
$a = b = c \Rightarrow z^2 = x^2 + y^2$

$z = \pm\sqrt{x^2 + y^2}$

2. $z = \sqrt{x^2 + y^2}$



3. $z = -\sqrt{x^2 + y^2}$



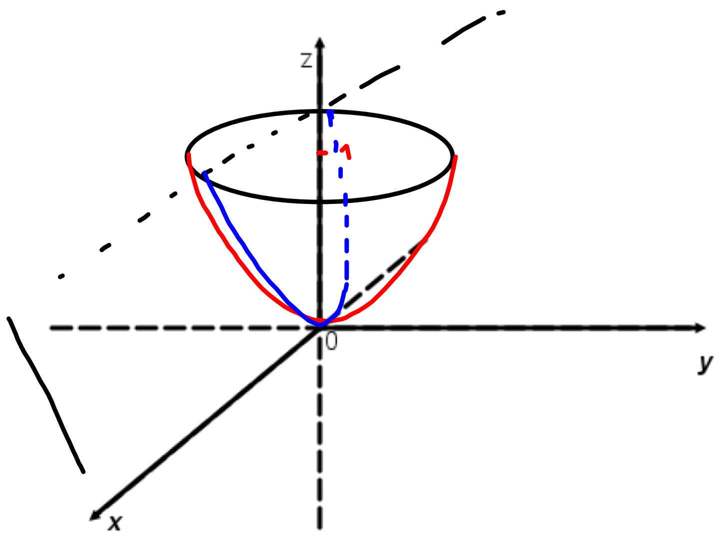
- Paraboloids There are two types:
  - *Elliptic paraboloid.* Standard equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

EXAMPLE 5. *Sketch the elliptic paraboloid*

$$z = \frac{x^2}{4} + \frac{y^2}{9} \geq 0$$

Plane	Trace	
$z = 1$	$\frac{x^2}{4} + \frac{y^2}{9} = 1$	ellipse
$x = 0$	$z = \frac{y^2}{9}$	parabola
$y = 0$	$z = \frac{x^2}{4}$	parabola



Special case: $a = b$

$z = \frac{c}{a^2}(x^2 + y^2)$

For example

$z = x^2 + y^2$

circular paraboloid

– *Hyperbolic paraboloid.* Standard equation:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$$

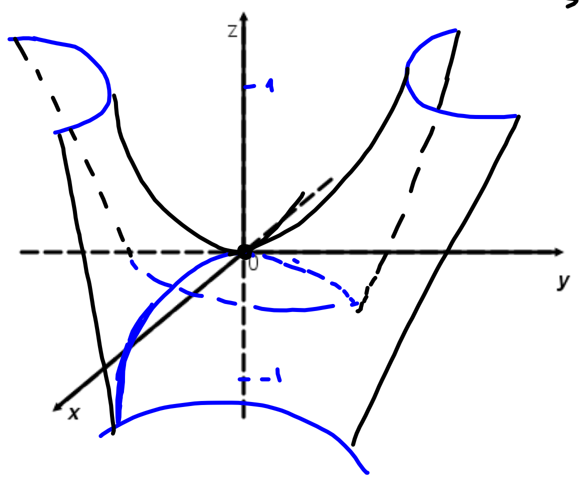
If $z = k$ then $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = \dfrac{k}{c}$

EXAMPLE 6. *Sketch the hyperbolic paraboloid*

$$z = x^2 - y^2$$

Plane	Trace
$z = 1$	$x^2 - y^2 = 1$ hyperbola
$z = -1$	$x^2 - y^2 = -1$ hyperbola
$x = 0$	$z = -y^2$ parabola
$y = 0$	$z = x^2$ parabola

saddle



The solution is posted in key for Quiz 3!

- Quadric cylinders: There are three types:

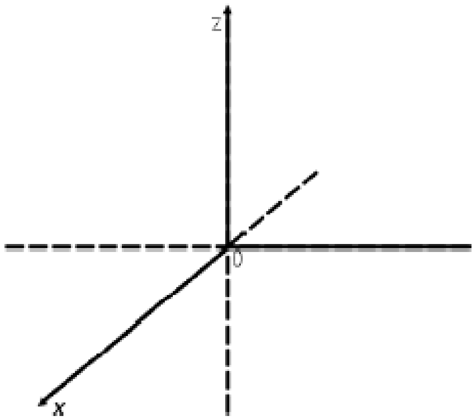
Elliptic cylinder:

- Standard equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

EXAMPLE 7. *Sketch elliptic cylinder*

$$x^2 + \frac{y^2}{4} = 1$$



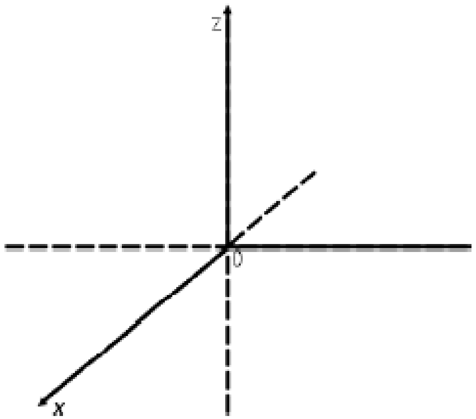
Hyperbolic cylinder:

- Standard equation:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

EXAMPLE 8. *Sketch hyperbolic cylinder*

$$x^2 - y^2 = 1$$



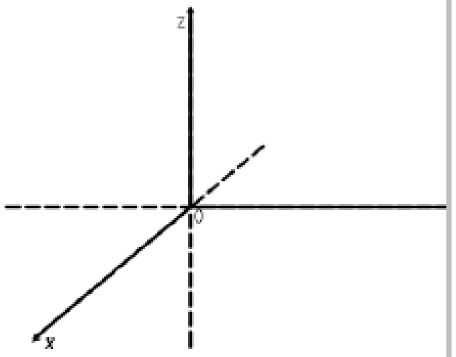
Parabolic cylinder:

- Standard equation:

$$y = ax^2$$

EXAMPLE 9. *Sketch parabolic cylinder*

$$y = -x^2$$



CONCLUSION

Ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
Hyperboloid of one sheet	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
Hyperboloid of two sheets	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
Elliptic Cones	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$
Elliptic paraboloid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$
Hyperbolic paraboloid	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$
Elliptic cylinder	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
Hyperbolic cylinder	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
Parabolic cylinder	$y = ax^2$

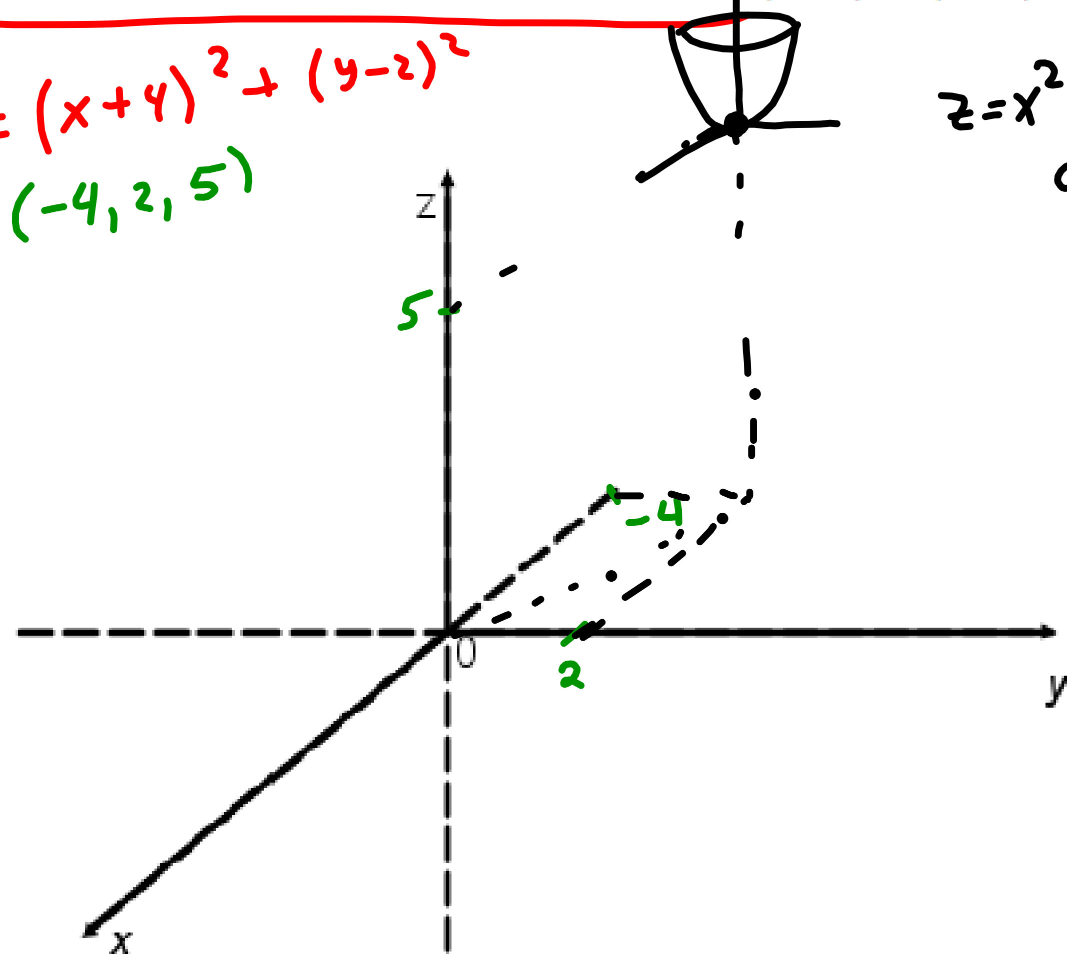
EXAMPLE 10. *Describe and sketch the surface* $z = (x+4)^2 + (y-2)^2 + 5$.

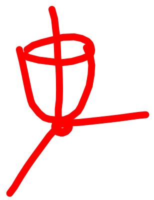
$z - 5 = (x+4)^2 + (y-2)^2$

vertex $(-4, 2, 5)$

$z = x^2 + y^2$

circular paraboloid



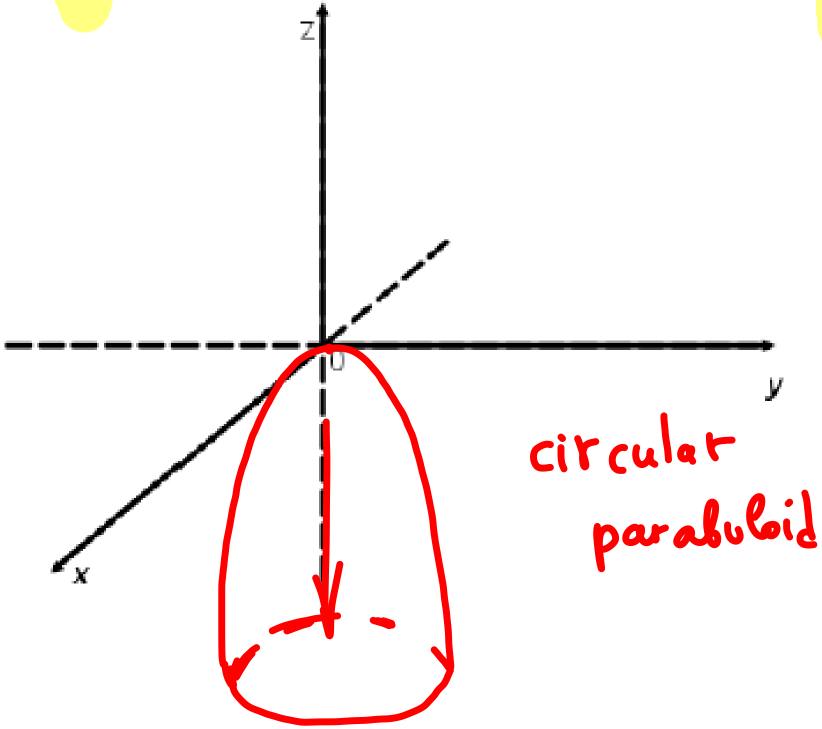


Note that **replacing a variable by its negative in the equation of a surface causes that surface to be reflected about a coordinate plane.**

EXAMPLE 11. *Identify and sketch the surface.*

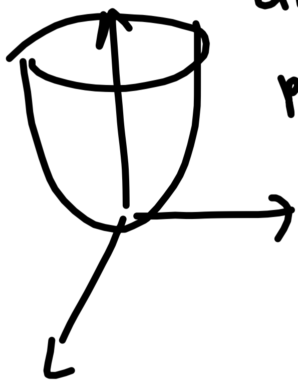
(a) $z = -(x^2 + y^2)$

$-z = x^2 + y^2$



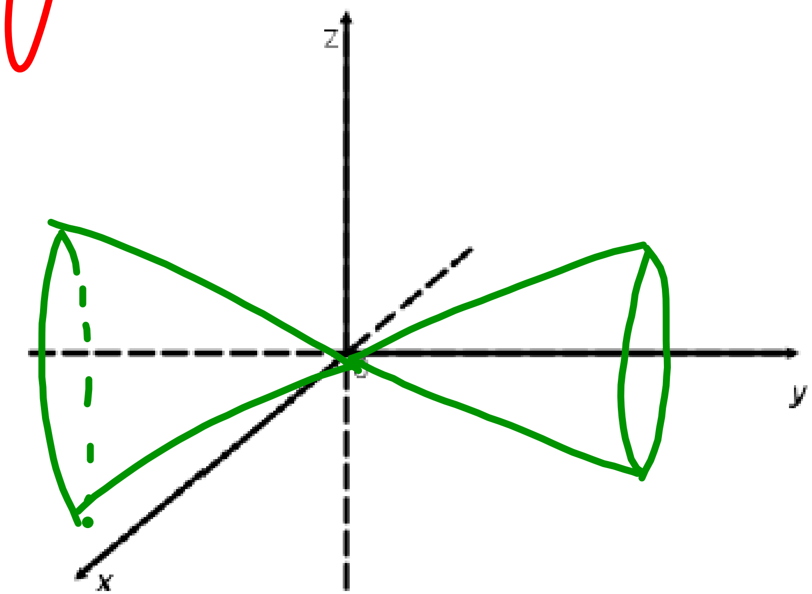
circular paraboloid

We know the graph of $z = x^2 + y^2$



circular paraboloid

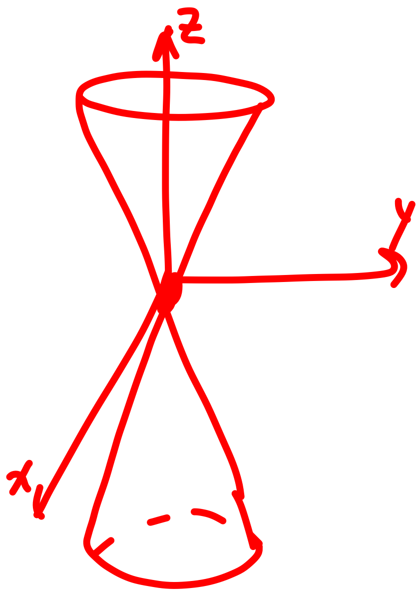
(b) $y^2 = x^2 + z^2$ circular cone



Compare with

$z^2 = x^2 + y^2$

circular cone



EXAMPLE 12. *Classify and sketch the surface*

$$x^2 + y^2 + z - 4x - 6y + 13 = 0.$$

Complete square : $(a \pm b)^2 = a^2 \pm 2ab + b^2$

$$\underbrace{x^2 - 4x + 4}_{(x-2)^2} - \cancel{4} + \underbrace{y^2 - 6y + 9}_{(y-3)^2} - \cancel{9} + z + \cancel{13} = 0$$

Compare to

$z = x^2 + y^2$

circular paraboloid

$$-z = (x-2)^2 + (y-3)^2$$

vertex $(2, 3, 0)$