12.2: Continuity

DEFINITION 1. *A function $f(x, y)$ is continuous at the point (x_0, y_0) if*

$$\lim_{(x,y)\to(x_0,y_0)} = f(x_0, y_0).$$

Roughly speaking, a function will be continuous at a point if the graph does not have any holes or breaks at that point.

All the standard functions that we know to be continuous are still continuous even if we are plugging in more than one variable now. We just need to watch out for division by zero, square roots of negative numbers, logarithms of zero or negative numbers, etc.

FACT: A polynomial of function of (x, y) is continuous. For example,

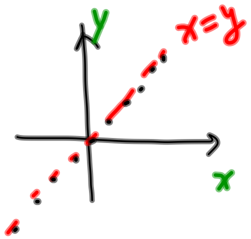
$$f(x,y) = x^5 + x^3 y + 5 - 3x$$

FACT: A composition of continuous functions is continuous. For example,

$$f(x,y) = \sin(xy) + e^{\cos(x-y)}$$

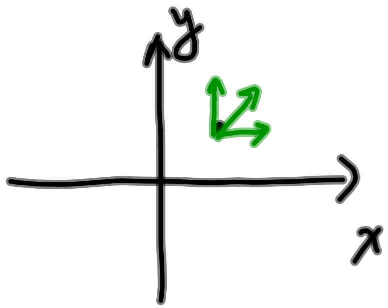
Examples of functions which are not continuous:

$$f(x,y) = \frac{1}{x-y}$$



not continuous when $x = y$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



12.3: Partial Derivatives

DEFINITION 2. *If f is a function of two variables, its* **partial derivatives** *are the functions f_x and f_y defined by*

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Conclusion: $f_x(x,y)$ represents the *rate of change* of the function $f(x,y)$ as we change x and hold y fixed while $f_y(x,y)$ represents the rate of change of $f(x,y)$ as we change y and hold x fixed.

Notations for partial derivatives: If $z = f(x,y)$, we write

$\partial \quad \partial \neq d$

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(f(x,y)) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

RULE FOR FINDING PARTIAL DERIVATIVES OF $z = f(x,y)$:

1. To find f_x, regard y as a constant and differentiate $f(x,y)$ with respect to x.

2. To find f_y, regard x as a constant and differentiate $f(x,y)$ with respect to y.

EXAMPLE 3. *If $f(x,y) = x^3 + y^5 e^x$ find $f_x(0,1)$ and $f_y(0,1)$.*

$$f_x(x,y) = 3x^2 + y^5 e^x \Rightarrow f_x(0,1) = 3 \cdot 0^2 + 1^5 e^0 = 1$$

$$f_y(x,y) = 0 + 5y^4 e^x \Rightarrow f_y(0,1) = 5$$

EXAMPLE 4. *Find all of the first order partial derivatives for the following functions:*

(a) $z(x, y) = x^3 \sin(xy)$

$$z_x \overset{PR}{=} (x^3)' \sin xy + x^3 \underbrace{\frac{\partial}{\partial x}(\sin xy)}_{y\cos(xy)} = 3x^2 \sin(xy) + x^3 y \cos(xy)$$

$$z_y = x^3 \frac{\partial}{\partial y}(\sin(xy)) = x^3 x \cos(xy) = x^4 \cos(xy)$$

(c) $u(x, y, z) = ye^{xyz}$

$$u_x = \frac{\partial}{\partial x}(ye^{xyz}) = y \frac{\partial}{\partial x}(e^{xyz}) = y e^{xyz} \underbrace{\frac{\partial}{\partial x}(xyz)}_{yz} = y^2 z e^{xyz}$$

$$u_y = \frac{\partial}{\partial y}(y) e^{xyz} + y \frac{\partial}{\partial y}(e^{xyz})$$

$$= e^{xyz} + y e^{xyz} xz$$

$$u_z = y \frac{\partial}{\partial z}(e^{xyz}) = y e^{xyz} \frac{\partial}{\partial z}(xyz) = y e^{xyz} xy$$

$$= xy^2 e^{xyz}$$

EXAMPLE 5. *The temperature at a point* (x, y) *on a flat metal plate is given by*

$$T(x,y) = \frac{80}{1+x^2+y^2}, = 80(1+x^2+y^2)^{-1}$$

where T *is measured in* °C *and* x, y *in meters. Find the rate of change of temperature with respect to distance at the point* $(1, 2)$ *in the* y*-direction.*

$$\frac{\partial T}{\partial y} = -\frac{80}{(1+x^2+y^2)^2}\cdot\frac{\partial}{\partial y}(1+x^2+y^2) = \frac{-80}{(1+x^2+y^2)^2}\cdot 2y$$

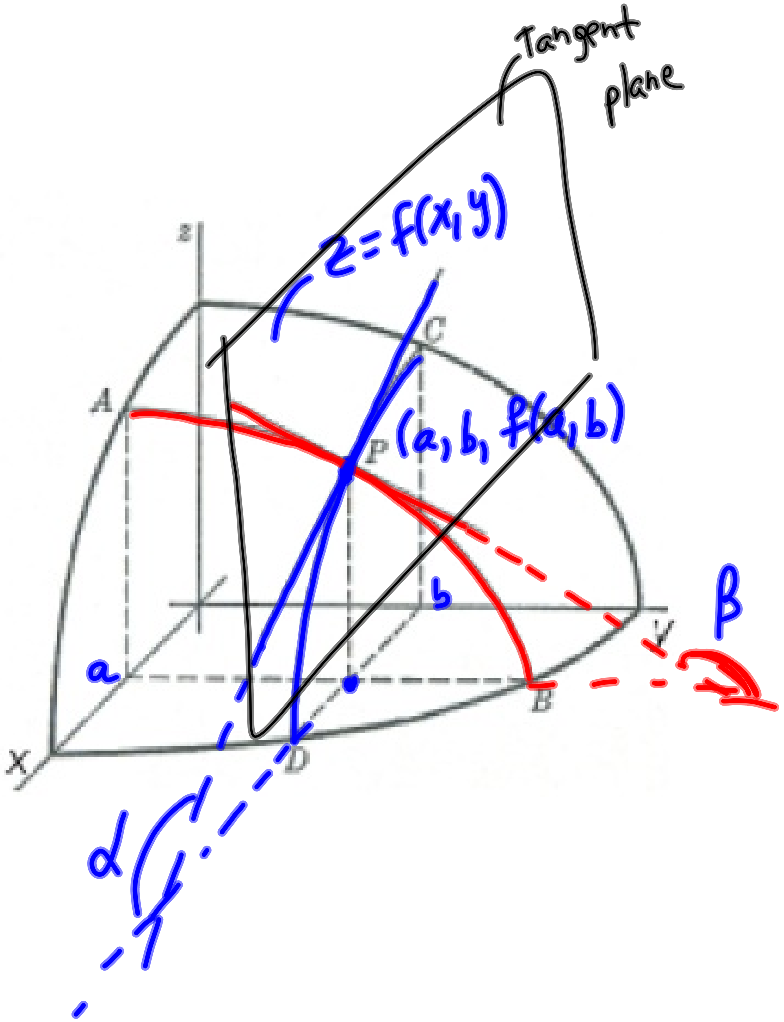
$$\frac{\partial T(1,2)}{\partial y} = -\frac{80\cdot 2\cdot 2}{(1+1^2+2^2)^2} = -\frac{80\cdot \cancel{4}^{1}}{\cancel{36}_{9}} = -\frac{80}{9}\ {}^\circ C/m$$

decreasing

Geometric interpretation of partial derivatives: Partial derivatives are the *slopes of traces*:

- $f_x(a,b)$ is the slope of the trace of the graph of $z = f(x,y)$ for the plane $y = b$ at the point (a,b).

$f_x(a,b) = \tan\alpha$ (slope)



- $f_y(a,b)$ is the slope of the trace of the graph of $z = f(x,y)$ for the plane $x = a$ at (a,b).

$f_y(a,b) = \tan\beta$ (slope)

$$(\sqrt{u})' = \frac{1}{2\sqrt{u}}$$

EXAMPLE 6. *If $f(x,y) = \sqrt{4 - x^2 - 4y^2}$, find $f_x(1,0)$ and $f_y(1,0)$ and interpret these numbers as slopes. Illustrate ~~with sketches~~.*

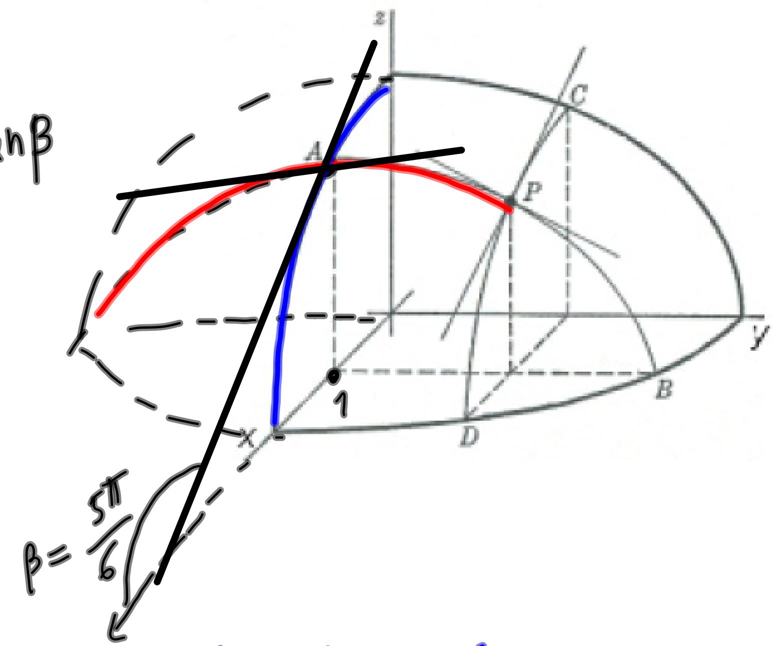
$\to$ the graph $z = \sqrt{4-x^2-4y^2}$

$$z^2 = 4 - x^2 - 4y^2, \quad z \geq 0$$

$$x^2 + 4y^2 + z^2 = 4, \quad z \geq 0 \text{ (ellipsoid)}$$

$$f_x = \frac{1}{2\sqrt{4-x^2-4y^2}} \cdot (-2x)$$

$$f_x(1,0) = -\frac{2}{2\sqrt{4-1}} = -\frac{1}{\sqrt{3}} = \boxed{-\frac{\sqrt{3}}{3}} = \tan\beta$$



$$f_y = \frac{1}{2\sqrt{4-x^2-4y^2}} \cdot (-8y)$$

$f_y(1,0) = \boxed{0} \Rightarrow$ tangent line in y-direction is horizontal

Higher derivatives: Since both of the first order partial derivatives for $f(x,y)$ are also functions of x and y, so we can in turn differentiate each with respect to x or y. We use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$\text{mixed part. derivatives}\left\{\begin{aligned} (f_x)_y &= f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x} \\ (f_y)_x &= f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y} \end{aligned}\right.$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

EXAMPLE 7. *Find the second partial derivatives of*

$$f(x,y) = y^3 + 5y^2e^{4x} - \cos(x^2).$$

1st order

$$f_x = 0 + 5y^2 \cdot 4e^{4x} + 2x\sin(x^2) = 20y^2e^{4x} + 2x\sin(x^2)$$

$$f_y = 3y^2 + 10ye^{4x}$$

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(20y^2e^{4x} + 2x\sin(x^2)\right) = 20y^2 \cdot 4e^{4x} + 2\sin x^2 + 2x \cdot 2x\cos x^2$$

$$f_{yy} = (f_y)_y = 6y + 10e^{4x}$$

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(20y^2e^{4x} + \underbrace{2x\sin x^2}_{\to 0}\right) = 40ye^{4x}$$

$$//$$

$$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\underbrace{3y^2}_{\to 0} + 10ye^{4x}\right) = 40ye^{4x}$$

We got $f_{xy} = f_{yx}$ ☺

Clairaut's Theorem. *Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D then*

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Partial derivative of order three or higher can also be defined. For instance,

3rd order

$$f_{yyx} = (f_{yy})_x = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y^2}\right) = \frac{\partial^3 z}{\partial x \partial y^2}.$$

Using Clairaut's Theorem one can show that if the functions f_{yyx}, f_{xyy} and f_{yxy} are continuous then

$$f_{yyx} = f_{xyy} = f_{yxy}$$

$$\neq$$

$$f_{xxy}$$

EXAMPLE 8. *Find the indicated derivative for*

$$f(x, y, z) = \cos(xy + z).$$

continuous with cont. partial derivatives of any order

(a) f_{xy}

$$f_x = -\sin(xy+z)\frac{\partial}{\partial x}(xy+z) = -y\sin(xy+z)$$

$$f_{xy} = \frac{\partial}{\partial y}(f_x) = -\frac{\partial}{\partial y}\left(y\sin(xy+z)\right) \overset{P.R.}{=} -\left[\sin(xy+z) + y\cos(xy+z)x\right]$$

(b) f_{zxy}

$$f_{zxy} = f_{xyz} = \left(f_{xy}\right)_z = -\left[\cos(xy+z) - xy\sin(xy+z)\right]$$

EXAMPLE 9. *If f and g are twice differentiable functions of a single variable, show that the function*

$$u(x,t) = f(x+at) + g(x-at)$$

is a solution of the wave equation

$$u_{tt} = a^2 u_{xx}.$$

See WIR #3