12.5: The Chain Rule

Chain Rule for functions of a single variable: If $y = f(x)$ and $x = g(t)$ where f and g ar differentiable functions, then y is indirectly a differentiable function of t and

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dg}{dt}.$$

EXAMPLE 1. *Let $z = x^y$, where $x = t^2$, $y = \sin t$. Compute $z'(t)$.*

$$z = x^y = (t^2)^{\sin t}$$

$$\frac{dz}{dt} = ?$$

Doesn't work

$$(x^n)' = n x^{n-1}$$

$$(a^x)' = a^x \ln a$$

Use Logarithmic Differentiation

$$z = (t^2)^{\sin t}$$

$$\ln z = \ln (t^2)^{\sin t} \Rightarrow \frac{d}{dt}(\ln z) = \frac{d}{dt}(\sin t \ln t^2)$$

$$\frac{z'}{z} = \cos t \ln t^2 + \frac{2}{t}\sin t$$

$$z' = (\cos t \ln t^2 + \frac{2}{t}\sin t)\cdot z$$

$$\boxed{z' = (\cos t \ln t^2 + \frac{2}{t}\sin t)x^y}$$

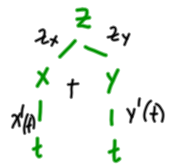
$$z' = (\cos t \ln t^2 + \frac{2}{t}\sin t)(t^2)^{\sin t}$$

Assume that all functions below have continuous derivatives (ordinary or partial).

- CASE 1: $z = f(x,y)$, where $x = x(t)$, $y = y(t)$ and compute $z'(t)$.

  Chain Rule:
  $$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$
  $$z_x x' + z_y y'$$

TREE DIAGRAM

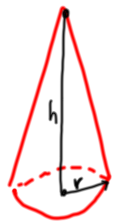


SOLUTION OF EXAMPLE 1:

Let $z = x^y$, *where* $x = t^2$, $y = \sin t$. *Compute* $z'(t)$.

$$\frac{dz}{dt} = z_x x' + z_y y' = y x^{y-1} \cdot 2t + x^y \ln x \cos t$$
$$= \sin t \, (t^2)^{\sin t - 1} \cdot 2t + (t^2)^{\sin t} \ln t^2 \cos t$$

EXAMPLE 2. *The radius of a right circular cone is increasing at a rate of* 1.8 *cm/s while its height is decreasing at a rate* 2.5 *cm/s. At what rate is the volume of the cone changing when the radius is* 120 *cm and the height is* 140 *cm.*



$$V = \frac{1}{3}\pi r^2 h = V(r,h)$$

$$r = r(t) \quad , \quad h = h(t)$$

$$\frac{dr}{dt} = 1.8 \text{ cm/s} \qquad \frac{dh}{dt} = -2.5 \text{ cm/s}$$

Find $\left.\dfrac{dV}{dt}\right|_{\substack{r=120 \\ h=140}}$

Apply Chain Rule

$$V(r(t), h(t)) = \frac{1}{3}\pi [r(t)]^2 \cdot h(t)$$

$$\frac{dV}{dt} = V_r \cdot r' + V_h \cdot h'$$

$$\frac{dV}{dt} = \frac{2}{3}\pi r h \cdot r' + \frac{1}{3}\pi r^2 h'$$

$$\left.\frac{dV}{dt}\right|_{\substack{r=120 \\ h=140}} = \frac{2}{3}\pi \cdot 120 \cdot 140 \cdot 1.8 + \frac{1}{3}\pi (120)^2 \cdot (-2.5)$$

$$= \frac{\pi}{3} 100 \left[\cancel{2} \cdot 12 \cdot 14 \cdot \frac{18}{\cancel{10}_{5}} - 12^2 \cdot \frac{\cancel{10}^{5}}{\cancel{4}_{2}} \right] = \ldots.$$

$$\frac{\pi}{3} \cdot 100 \cdot 12 \left[\frac{14 \cdot 18}{5} - 6 \cdot 5 \right] = \ldots$$

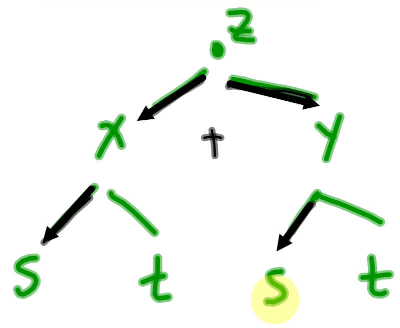
- CASE 2: $z = f(x, y)$, where $x = x(s,t)$, $y = y(s,t)$ and compute z_s and z_t.

Chain Rule:

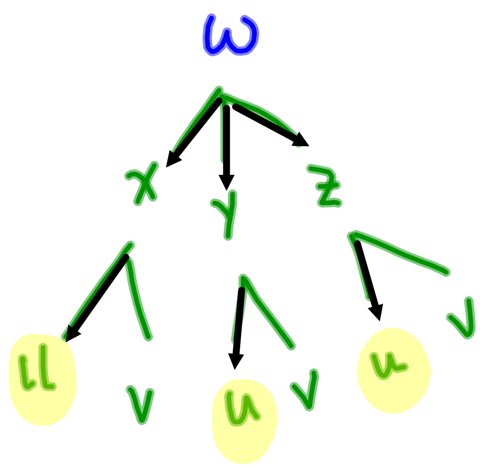
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Tree diagram:



EXAMPLE 3. *Write out the Chain Rule for the case where $w = f(x, y, z)$ and $x = x(u, v)$, $y = y(u, v)$ and $z = z(u, v)$.*



$$w = f(x,y,z) = f(x(u,v),\, y(u,v),\, z(u,v))$$

$$\frac{\partial w}{\partial u} = w_x x_u + w_y y_u + w_z z_u$$

$$\frac{\partial w}{\partial v} = w_x x_v + w_y y_v + w_z z_v$$

EXAMPLE 4. *If* $z = \sin x \cos y$, *where* $x = (s-t)^2$, $y = s^2 - t^2$ *find* $z_s + z_t$.

$$z_s = z_x x_s + z_y y_s = \cos x \cos y \, 2(s-t) - \sin x \sin y \, 2s$$

$$+ \quad z_t = z_x x_t + z_y y_t = \cos x \cos y \, (-2(s-t)) - \sin x \sin y \, (-2t)$$

$$z_s + z_t = 0 - 2 \sin x \sin y \, (s-t)$$

$$= -2(s-t) \sin x \sin y$$

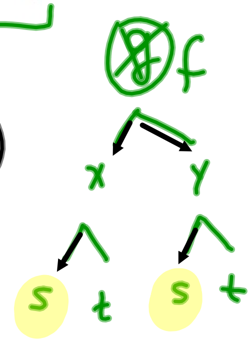
$$= -2(s-t) \sin (s-t)^2 \sin (s^2 - t^2)$$

EXAMPLE 5. *Show that*

$$g(s,t) = f(\overbrace{s^2 - t^2}^{x}, \overbrace{t^2 - s^2}^{y})$$

satisfies the equation

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0.$$



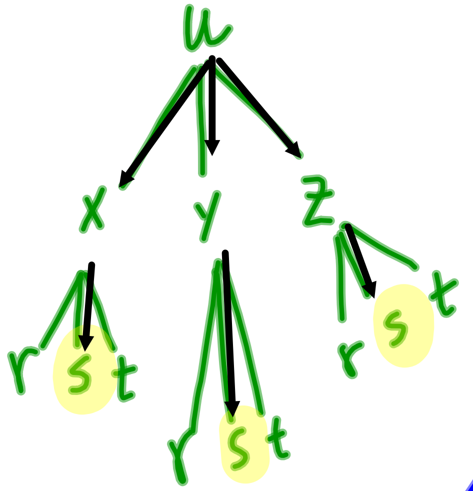
$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial s} = f_x \cdot x_s + f_y \cdot y_s = f_x \cdot 2s + f_y \cdot (-2s)$$

$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial t} = f_x\, x_t + f_y\, y_t = f_x(-2t) + f_y \cdot 2t$$

$$t\frac{\partial g}{\partial s} + s\left(\frac{\partial g}{\partial t}\right) = t\left(2s f_x - 2s f_y\right) + s\left(-2t f_x + 2t f_y\right)$$

$$= 2st\, f_x - 2st\, f_y - 2ts\, f_x + 2st\, f_y = 0$$

EXAMPLE 6. *If* $u = x^2y + y^3z^2$ *where* $x = rse^t$, $y = r + s^2e^{-t}$, $z = rs\sin t$, *find* u_s *when* $(r, s, t) = (1, 2, 0)$



$$u_s = u_x x_s + u_y y_s + u_z z_s$$

$$u_s = 2xy\, re^t + (x^2 + 3y^2z^2)\, 2se^{-t} + 2y^3z\, rs\sin t$$

When $(r, s, t) = (1, 2, 0)$ we have

$$x = (rse^t)\big|_{(1,2,0)} = 1\cdot 2e^0 = 2$$

$$y = (r + s^2e^{-t})\big|_{(1,2,0)} = 1 + 4e^0 = 5$$

$$z = (rs\sin t)\big|_{(1,2,0)} = 0$$

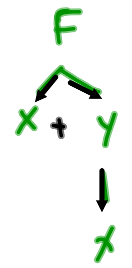
$$u_s\big|_{(1,2,0)} = 2\cdot 2\cdot 5\cdot 1\cdot e^0 + (2^2 + 3\cdot 5^2\cdot 0)\cdot 2\cdot 2e^0 + 2\cdot 5^3\cdot 0\cdot 1\cdot \sin 0 = 20 + 16 = \boxed{36}$$

Implicit differentiation: Suppose that an equation

$$F(x, y) = 0$$

defines y implicitly as a differentiable function of x, i.e. $y = y(x)$, where $F(x, y(x)) = 0$ for all x in the domain of $y(x)$. Find y':

$$F(x,y)=0 \quad \Rightarrow \quad \frac{d}{dx} F(x, y(x)) = \frac{d}{dx} 0$$



$$F_x + F_y \cdot y' = 0$$

$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}}$$

$$F_y\, y' = -F_x \quad \Rightarrow \quad \boxed{y' = -\frac{F_x}{F_y}}$$

EXAMPLE 7. *Find y' if $\underline{x^4 + y^3 = 6e^{xy}}$.* Implicit function

$$\underbrace{F(x,y)} = x^4 + y^3 - 6e^{xy} = 0$$

$$\left.\begin{aligned} F_x &= 4x^3 - 6ye^{xy} \\ F_y &= 3y^2 - 6xe^{xy} \end{aligned}\right\} \Rightarrow y' = -\frac{F_x}{F_y} = -\frac{4x^3 - 6ye^{xy}}{3y^2 - 6xe^{xy}}$$

Suppose that an equation

$$F(x, y, z) = 0$$

defines z implicitly as a differentiable function of x and y, i.e. $z = z(x, y)$, where

$$\frac{\partial}{\partial x} F(x, y, z(x, y)) = \frac{\partial}{\partial x} 0$$

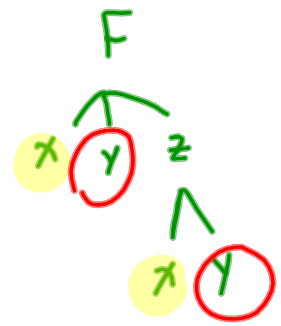
for all (x, y) in the domain of z. Find the partial derivatives z_x and z_y:

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}(0)$$

$$\Downarrow$$

$$\frac{\partial F}{\partial x} = 0$$

$$F_x + F_z \cdot z_x = 0$$



$$F_z \cdot z_x = -F_x$$

$$z_x = -\frac{F_x}{F_z} \Rightarrow \boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}}$$

$$F_y + F_z \cdot z_y = 0$$

$$z_y = -\frac{F_y}{F_z}$$

$$\boxed{\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}}$$

EXAMPLE 8. *If* $x^4 + y^3 + z^2 + xye^z = 10$ *find*

(a) z_x *and* z_y

$$F(x,y,z) = x^4 + y^3 + z^2 + xye^z - 10 = 0$$

$$F_x = 4x^3 + ye^z$$

$$F_y = 3y^2 + xe^z$$

$$F_z = 2z + xye^z$$

$$z_x = \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{4x^3 + ye^z}{2z + xye^z}$$

$$z_y = \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2 + xe^z}{2z + xye^z}$$

(b) x_y *and* x_z

$$x_y = \frac{\partial x}{\partial y} = -\frac{F_y}{F_x} = -\frac{3y^2 + xe^z}{4x^3 + ye^z}$$

$$x_z = \frac{\partial x}{\partial z} = -\frac{F_z}{F_x} = -\frac{2z + xye^z}{4x^3 + ye^z}$$