12.6: Directional Derivatives and the Gradient Vector

Recall that the two partial derivatives $f_x(x,y)$ and $f_y(x,y)$ of $f(x,y)$ represent the rate of change of f as we vary x (holding y fixed) and as we vary y (holding x fixed) respectively. In other words, $f_x(x,y)$ and $f_y(x,y)$ represent the rate of change of f in the directions of the unit vectors **i** and **j** respectively. Let's consider how to find the rate of change of f if we allow both x and y to change simultaneously, or how to find the rate of change of f in the direction of an arbitrary vector **u**.

DEFINITION 1. *The rate of change of $f(x,y)$ in the direction of the unit vector $\hat{\mathbf{u}} = \langle a,b\rangle$ is called the* **directional derivative** *and it is denoted by $D_{\mathbf{u}}f(x,y)$.*

The **directional derivative** *of f at (x_0,y_0) in the direction of the unit vector $\hat{\mathbf{u}} = \langle a,b\rangle$ is*

$$D_{\mathbf{u}}f(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}$$

if this limit exists.

REMARK 2. By comparing the last definition with the definitions of the partial derivatives:

$$f_x(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}, \quad f_y(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0,y_0+h)-f(x_0,y_0)}{h}$$

we see that

$f_y(x_0,y_0) = D_{\vec{j}}f(x_0,y_0)$ and $f_x(x_0,y_0) = D_{\vec{i}}f(x_0,y_0)$

For computational purposes use the following theorem.

means that there is tangent plane at (x,y)

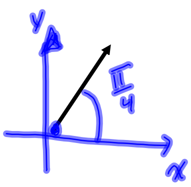
THEOREM 3. *If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\hat{\mathbf{u}} = \langle a,b\rangle$ and*

$a^2+b^2=1$

$$D_{\mathbf{u}}f(x,y) = f_x(x,y)a + f_y(x,y)b.$$

EXAMPLE 4. *Find the rate of change $f(x,y) = x^3 + \sin(xy)$ at the point $(1,\pi/2)$ in the direction indicated by the angle $\theta = \pi/4$.*

'' directional derivative



$\hat{u} = \langle \cos\frac{\pi}{4}, \sin\frac{\pi}{4}\rangle = \langle \underbrace{\frac{\sqrt{2}}{2}}_{a}, \underbrace{\frac{\sqrt{2}}{2}}_{b}\rangle$

$f_x = 3x^2 + y\cos(xy)$

$f_y = x\cos(xy)$

$f_x(1,\frac{\pi}{2}) = 3 + \frac{\pi}{2}\cos\frac{\pi}{2} = 3$

$f_y(1,\frac{\pi}{2}) = 0$

$D_{\vec{u}}f(1,\frac{\pi}{2}) = f_x(1,\frac{\pi}{2})a + f_y(1,\frac{\pi}{2})b$

$= 3\cdot\frac{\sqrt{2}}{2} + 0\cdot\frac{\sqrt{2}}{2} = \boxed{\frac{3\sqrt{2}}{2}}$

The Directional Derivative As The Dot Product Of Two Vectors. Gradient.

DEFINITION 5. *The* **gradient** *of* $f(x, y)$ *is the vector function* ∇f *defined by*

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

Notations for gradient: **grad**f or ∇f which is read "del f". nabla

EXAMPLE 6. *Find the gradient of* $f = \cos(xy) + e^x$ *at (0,3).*

$$f_x = -y\sin(xy) + e^x \Big|_{(0,3)} = 0 + e^0 = 1$$

$$f_y = -x\sin(xy) \Big|_{(0,3)} = 0$$

$$\nabla f(0,3) = \langle 1, 0\rangle = \hat{i}$$

By Theorem 3 we have:

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b = \langle f_x, f_y\rangle \cdot \langle a, b\rangle$$

`Formula for the directional derivative using the gradient vector:`

$$\boxed{D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \hat{\mathbf{u}}.}$$

unit

EXAMPLE 7. *Find the directional derivative for* f *from Example 6 at* $(0, 3)$ *in the direction* $\langle 3, 4\rangle$.

The directional derivative of function of *three* variables

THEOREM 8. *If f is a differentiable function of x, y and z, then f has a directional derivative in the direction of any unit vector $\hat{\mathbf{u}} = \langle a, b, c\rangle$ and*

$$D_{\mathbf{u}}f(x,y) = f_x(x,y,z)a + f_y(x,y,z)b + f_z(x,y,z)c = \underbrace{\nabla f \cdot \hat{\mathbf{u}}},$$

where

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

is the gradient vector of $f(x,y,z)$.

EXAMPLE 9. *Find the directional derivative of $f(x,y,z) = z^3 - x^2y$ at the point $(1,6,2)$ in the direction $\mathbf{u} = \langle 1,-2,3\rangle$.*

$$\left.\begin{aligned} f_x &= -2xy\Big|_{(1,6,2)} = -12 \\ f_y &= -x^2\Big|_{(1,6,2)} = -1 \\ f_z &= 3z^2\Big|_{(1,6,2)} = 12 \end{aligned}\right\} \Rightarrow \nabla f(1,6,2) = \langle -12,-1,12\rangle$$

$$\left.\begin{aligned} &\text{Normalize } \hat{u} = \frac{\vec{u}}{|\vec{u}|} \\ &|\vec{u}| = \sqrt{1^2+4+9} = \sqrt{14} \end{aligned}\right\} \Rightarrow \hat{u} = \frac{\langle 1,-2,3\rangle}{\sqrt{14}}$$

$$D_u f(1,6,2) = \nabla f(1,6,2)\cdot\hat{u} = \frac{\langle -12,-1,12\rangle\cdot\langle 1,-2,3\rangle}{\sqrt{14}}$$

$$= \frac{-12+2+36}{\sqrt{14}} = \boxed{\frac{26}{\sqrt{14}}}$$

QUESTION: In which of all possible directions does f change fastest and what is the maximum rate of change.

ANSWER is provided by the following theorem:

THEOREM 10. *Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}}f$ is $|\nabla f|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector ∇f.*

Proof. see Ex. 11 (Sec. 11.1 - 11.3)

$$\max_{\vec{u}} D_{\vec{u}} f = |\nabla f| = D_{\nabla f} f$$

EXAMPLE 11. *Suppose that the temperature at a point* (x, y, z) *in the space is given by*

$$T(x, y, z) = \frac{100}{1 + x^2 + y^2 + z^2},$$

where T *is measured in* $^\circ C$ *and* x, y, z *in meters.*

(a) *In which direction does the temperature increase fastest at the point* $(1, 1, -1)$*?*

In the direction $\nabla T(1,1,-1)$

$$T_x = -\frac{100 \cdot 2x}{(1+x^2+y^2+z^2)^2} \Bigg|_{(1,1,-1)} = -\frac{200 \cdot 1}{4^2} = -\frac{\cancel{200}^{\cancel{50}\,25}}{\cancel{16}_{\cancel{4}\,2}} = -\frac{25}{2}$$

$$T_y = -\frac{200y}{(\quad)^2} = -\frac{25}{2}$$

$$T_z = -\frac{200z}{(\quad)^2} = \frac{25}{2}$$

$$\nabla T(1,1,-1) = \boxed{\left\langle -\frac{25}{2}, -\frac{25}{2}, \frac{25}{2} \right\rangle}$$

(b) *What is the maximum rate of increase?*

$$|\nabla T(1,1,-1)| = \sqrt{3 \cdot \left(\frac{25}{2}\right)^2} = \boxed{\frac{25}{2}\sqrt{3}}$$

$u = f(x, y, z) \Rightarrow$ Level surfaces $f(x, y, z) = k$

Tangent planes to level surfaces:

$\perp$

FACT: The gradient vector $\nabla F(x_0, y_0, z_0)$ is **orthogonal** to the level surface $F(x, y, z) = k$ at the point (x_0, y_0, z_0).

$\vec{N}(x_0, y_0, z_0) = \nabla F(x_0, y_0, z_0) = \langle F_x(x_0, y_0, z_0), F_y(\ldots), F_z(\ldots) \rangle$

So, the *tangent plane* to the surface $f(x, y, z) = k$ at the point (x_0, y_0, z_0) has the equation

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

The normal line to the surface at the point (x_0, y_0, z_0) is the line passing through (x_0, y_0, z_0) and perpendicular to the tangent plane. Therefore its direction is given by the gradient vector

$\vec{v} = \nabla F(x_0, y_0, z_0)$

EXAMPLE 13. *Find the equation of the tangent plane and normal line at the point* $(1, 0, 5)$ *to the surface* $xe^{yz} = 1$.

(x_0, y_0, z_0)

$\vec{N} = \nabla F(1,0,5)$ where $F = xe^{yz}$

$\nabla F = \langle F_x, F_y, F_z \rangle = \langle e^{yz}, xze^{yz}, xye^{yz} \rangle$

$\vec{N} = \nabla F(1,0,5) = \langle 1, 5, 0 \rangle$

$1 \cdot (x-1) + 5(y-0) + 0(z-5) = 0$

$\boxed{x + 5y = 1}$ Tangent plane

Normal line through $(1,0,5)$ in the direction $\vec{v} = N = \langle 1, 5, 0 \rangle$

$$x = 1 + t \cdot 1$$
$$y = 0 + t \cdot 5$$
$$z = 5 + t \cdot 0$$

$\Rightarrow$

$$\boxed{\begin{aligned} x &= 1 + t \\ y &= 5t \\ z &= 5 \end{aligned}}$$

$z = f(x, y) \Rightarrow$ level curve $f(x,y) = k$

Likewise, the gradient vector $\nabla f(x_0, y_0)$ is **orthogonal** to the level curve $f(x, y) = k$ at the point (x_0, y_0).

Consider a topographical map of a hill and let $f(x, y)$ represent the height above sea level at a point with coordinates (x, y). Draw a curve of steepest ascent.