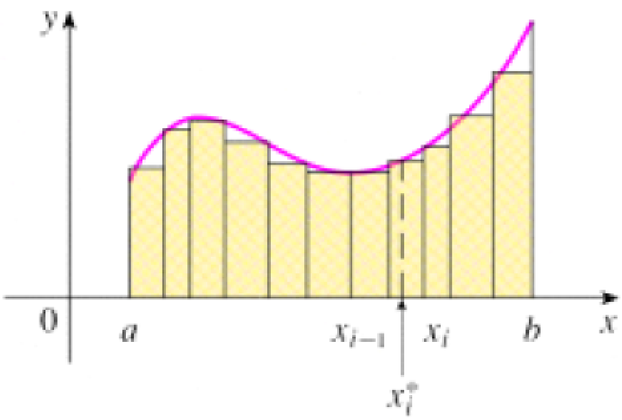
13.1: Double integrals over rectangles

Recall that a single definite integral can be interpreted as **area**:

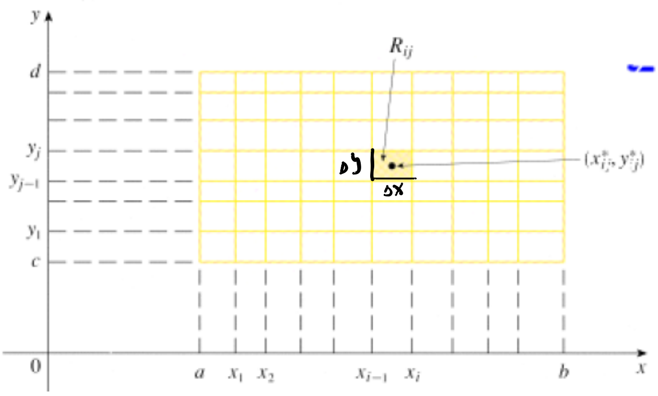


The exact area is also the definition of the definite integral:

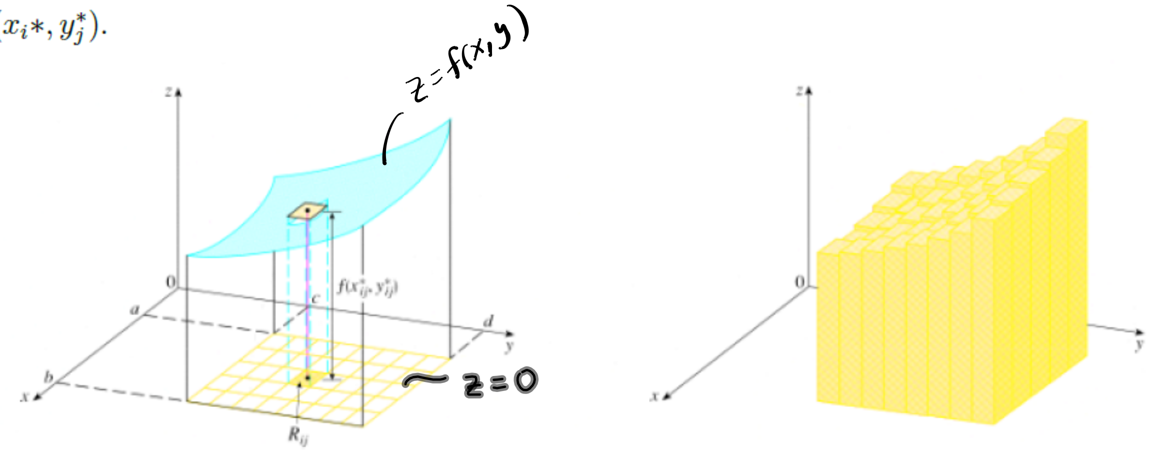
Area $(f \geq 0)$ $= \int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$

Problem: Assume that $f(x, y)$ is defined on a closed rectangle
$R = [a, b] \times [b, c] = \{(x, y) \in \mathbb{R}^2 | a \leq x \leq b, c \leq y \leq d\}$ and $f(x, y) \geq 0$ over R. Denote by S the part of the surface $z = f(x, y)$ over the rectangle R. What the volume of the region under S and above the xy-plane is?

Solution: Approximate the volume. Divide up $a \leq x \leq b$ into n subintervals and divide up $c \leq y \leq d$ into m subintervals. From each of these smaller rectangles choose a point (x_i^*, y_j^*).



Over each of these smaller rectangles we will construct a box whose height is given b $f(x_i*, y_j^*)$.



The volume is given by

$$\lim_{n,m\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{m}\underbrace{f(x_i^*,y_j^*)}_{\text{height}}\underbrace{\overset{= dA}{\Delta x\Delta y}}_{\text{base area}}$$

which is also the definition of a double integral

$$\iint_R f(x,y)\mathrm{d}A.$$

dA = dx dy for rectangle only

Another notation: $\displaystyle\iint_R f(x,y)\,\mathrm{d}A = \iint_R f(x,y)\,\mathrm{d}x\mathrm{d}y.$

THEOREM 1. *If f is continuous on R then f is integrable over R.*

THEOREM 2. *If $f(x,y)\geq 0$ and f is continuous on the rectangle $R=[a,b]\times[c,d]$, then the volume V of the solid S that lies above R and inder the graph of f, i.e.*

$$S=\left\{(x,y,z)\in\mathbb{R}^3|(x,y)\in R, 0\leq z\leq f(x,y),(x,y)\underbrace{\in}_{\text{in}} R\right\},$$

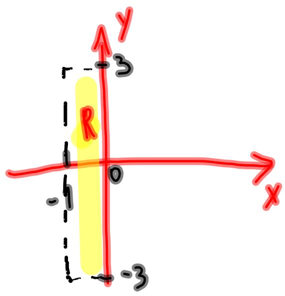
is

$$\boxed{V=\iint_R f(x,y)\,\mathrm{d}A.}$$

EXAMPLE 3. *Evaluate the integral*

$$\iint_R 4\,\mathrm{d}A$$ = volume of the box with base R and height = 4

where $R = [-1, 0] \times [-3, 3]$ *by identifying it as a volume of a solid.*



$z = f(x,y)$

$z = 4$

$$= 4 \cdot \text{Area}(R) = 4\,(6 \times 1) = \boxed{24}$$