# 13.2: Iterated integrals

Suppose that $f(x, y)$ is integrable over the rectangle $R = [a, b] \times [b, c]$.

**Partial integration of $f$ with respect to $x$:** $\displaystyle\int_a^b f(x, y)\,\mathrm{d}x$

**Partial integration of $f$ with respect to $y$:** $\displaystyle\int_c^d f(x, y)\,\mathrm{d}y$

EXAMPLE 1.

$$\int_0^4 (x + 3y^2)\,\mathrm{d}x = \frac{x^2}{2} + 3y^2 x \Big|_{x=0}^{4} = \frac{16}{2} + 3y^2 \cdot 4 - 0 = 8 + 12y^2$$

$$\int_1^4 e^{xy}\,\mathrm{d}y = \frac{1}{x} e^{xy} \Big|_{y=1}^{4} = \frac{1}{x}\left(e^{4x} - e^{x}\right)$$

*Iterated integrals:*

$$\int_a^b \left[\int_c^d f(x,y)\,\mathrm{d}y\right]\mathrm{d}x = \boxed{\int_a^b \int_c^d f(x,y)\,\mathrm{d}y\mathrm{d}x}$$

and

$$\int_c^d \left[\int_a^b f(x,y)\,\mathrm{d}x\right]\mathrm{d}y = \boxed{\int_c^d \int_a^b f(x,y)\,\mathrm{d}x\mathrm{d}y.}$$

EXAMPLE 2. *Evaluate the integrals:*

$$I_1 = \int_0^1 \int_1^4 x\sqrt{y}\,\mathrm{d}y\mathrm{d}x, \quad I_2 = \int_1^4 \int_0^1 x\sqrt{y}\,\mathrm{d}x\mathrm{d}y$$

$$I_1 = \int_0^1 \left[\int_1^4 x\sqrt{y}\,dy\right]dx = \int_0^1 x\left[\int_1^4 \sqrt{y}\,dy\right]dx = \left(\int_1^4 \sqrt{y}\,dy\right)\left(\int_0^1 x\,dx\right)$$

$$= \frac{\cancel{2}}{3} y^{3/2}\Big|_1^4 \cdot \frac{x^2}{\cancel{2}}\Big|_0^1$$

$$\frac{1}{3}\left[4^{3/2}-1\right]\cdot\left[1-0\right] = \boxed{7/3}$$

$I_1 = I_2$

FUBINI's THEOREM: *If $f$ is continuous on on the rectangle*

$$R = [a,b] \times [c,d]$$

*then*

double integral

iterated integrals

$$\underbrace{\iint_R f(x,y)\,\mathrm{d}A}_{\text{double integral}} = \int_a^b \int_c^d f(x,y)\,\mathrm{d}y\mathrm{d}x = \int_c^d \int_a^b f(x,y)\,\mathrm{d}x\mathrm{d}y.$$

EXAMPLE 3. *Evaluate*

$$\iint_R x \underbrace{\cos(xy)}_{\text{cont.}}\,\mathrm{d}A$$

*where* $R = \underbrace{[-\pi/2, \pi/2]}_{x} \times \underbrace{[1,5]}_{y}$

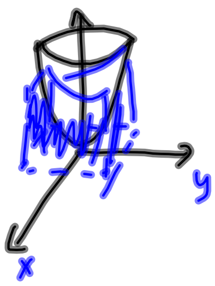
$$= \int_1^5 \int_{-\pi/2}^{\pi/2} x\cos(xy)\,dx\,dy = \underbrace{\int_{-\pi/2}^{\pi/2}\int_1^5 x\cos(xy)\,dy\,dx}_{\text{easier to find anti-derivative}}$$

easier to find anti-derivative

$$\int_{-\frac{\pi}{2}}^{\pi/2} x\left[\int_1^5 \underline{\cos(xy)}\,dy\right]dx = \int_{-\frac{\pi}{2}}^{\pi/2} \cancel{x}\,\frac{\sin(xy)}{\cancel{x}}\Bigg|_{y=1}^{5} dx = \int_{-\frac{\pi}{2}}^{\pi/2} \left(\sin(5x) - \sin x\right) dx$$

$$= -\frac{\cos 5x}{5} + \cos x \Bigg|_{-\pi/2}^{\pi/2} = 0$$

EXAMPLE 4. (~~see Section 12.1, Example 3~~) *Find the volume of the solid $S$ lying under the circular paraboloid* $\underbrace{z = x^2 + y^2}_{\text{graph}}$ *and above the rectangle* $R = [-2, 2] \times [-3, 3]$.




$$V = \iint_R \underbrace{(x^2+y^2)}_{\text{cont.}} \, dA = \int_{-2}^{2} \left[ \int_{-3}^{3} (x^2+y^2)\, dy \right] dx$$

$$= \int_{-2}^{2} x^2 y + \frac{y^3}{3} \Big|_{y=-3}^{3} dx = \int_{-2}^{2} 3x^2 + 9 - (-3x^2 - 9)\, dx$$

$$= \int_{-2}^{2} 6x^2 + 18\, dx = 2x^3 + 18x \Big|_{-2}^{2} = 2 \cdot 16 + 18 \cdot 4$$

$$= 32 + 72 = 104$$

rectangle

FACT: *If $g$ and $h$ are continuous functions of one variable and $R = [a, b] \times [c, d]$ then*

$$\iint_R g(x)h(y)\,\mathrm{d}A = \left(\int_a^b g(x)\mathrm{d}x\right)\left(\int_c^d h(y)\,\mathrm{d}y\right).$$

EXAMPLE 5. *If* $R = [\overset{x}{0, \ln 2}] \times [\overset{y}{0, \ln 5}]$ *find* $\iint_R e^{2x-y}\,\mathrm{d}A. = \iint_R e^{2x} e^{-y}\, dA =$

$$= \left(\int_0^{\ln 2} e^{2x}\,dx\right)\left(\int_0^{\ln 5} e^{-y}\,dy\right) = \frac{e^{2x}}{2}\Big|_0^{\ln 2} \cdot \left(-e^{-y}\right)\Big|_0^{\ln 5}$$

$$= \frac{1}{2}\left(e^{2\ln 2} - e^0\right)\cdot\left(-\left(e^{-\ln 5} - e^0\right)\right)$$

$$= -\frac{1}{2}\left(4 - 1\right)\left(\frac{1}{5} - 1\right) = \frac{1}{2}\cdot 3 \cdot \frac{4}{5} = \boxed{\frac{6}{5}}$$