13.3: Double integrals over general regions

All functions below are continuous on their domains.

$$\iint_D f(x,y)\, dA$$

Let D be a bounded region enclosed in a rectangular region R. We define

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not in } D. \end{cases}$$

If F is integrable over R, then we say F is *integrable* over D and we define **the double integral of f over D** by

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_R F(x,y)\,\mathrm{d}A = \underbrace{\iint_D F\, dA}_{=\iint_D f\, dA} + \underbrace{\iint_{R-D} F\, dA}_{=0}$$

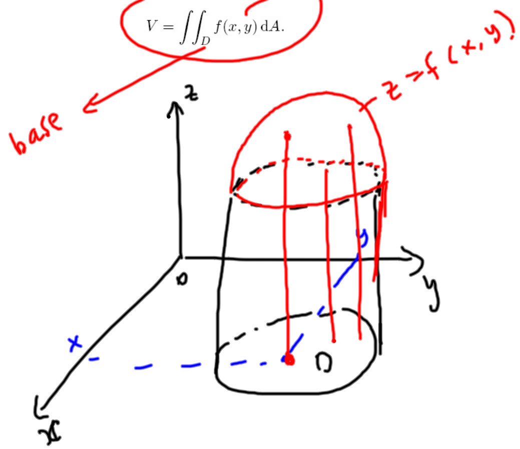
FACT: If $f(x,y) \geq 0$ and f is continuous on the region D then the volume V of the solid S that lies above D and under the graph of f, i.e.

$$\boxed{S = \{(x,y,z) \in \mathbb{R}^3 |\ 0 \leq z \leq f(x,y), (x,y) \in D\},}$$

LID

is

$$V = \iint_D f(x,y)\,\mathrm{d}A.$$



EXAMPLE 1. *Evaluate the integral*

$$\iint_D \overbrace{\sqrt{16 - x^2 - y^2}}^{\geq 0}\, \mathrm{d}A$$

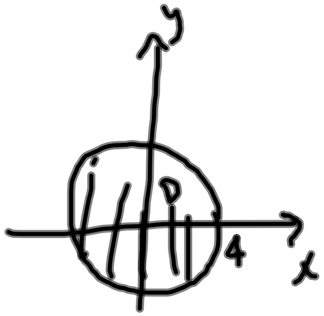
where $D = \{(x, y) \in \mathbb{R}^2 |\ x^2 + y^2 \leq 16\}$ *by identifying it as a volume of a solid.*

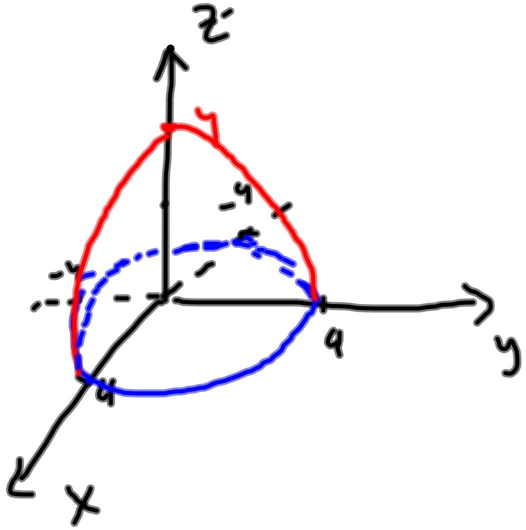
Lid: $z = \sqrt{16 - x^2 - y^2}$

$x^2 + y^2 + z^2 = 16,\ z \geq 0$

upper half sphere with $R = 4$

$$\iint_D f(x,y)\, dA$$





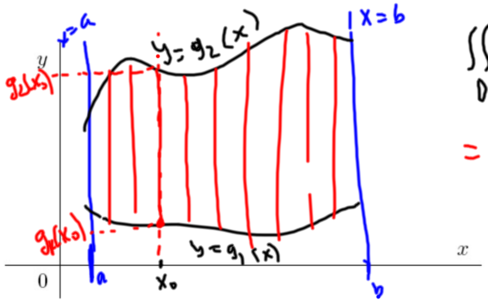
$$\iint_D \sqrt{16 - x^2 - y^2}\, dA = \frac{1}{2}\left(\text{Volume of ball with } R = 4\right)$$

$$= \frac{1}{2}\, \frac{4}{3}\pi R^3 = \frac{2}{3}\pi\, 4^3 = \frac{128\pi}{3}$$

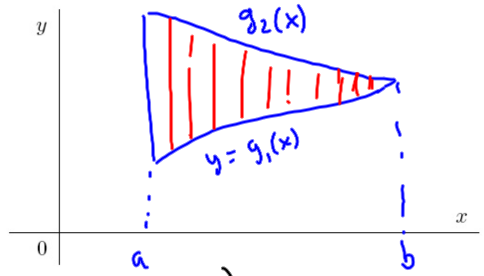
Computation of double integral:

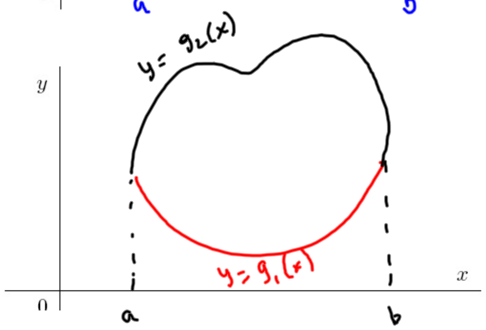
A plain region of **TYPE I**: (vertical)

$$D = \{(x, y)|\ a \le x \le b, g_1(x) \le y \le g_2(x)\}.$$



$$\iint_D f(x,y)\, dA = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right) dx$$





THEOREM 2. *If D is a region of type I such that* $D = \{(x, y)|\ a \le x \le b, g_1(x) \le y \le g_2(x)\}$ *then*

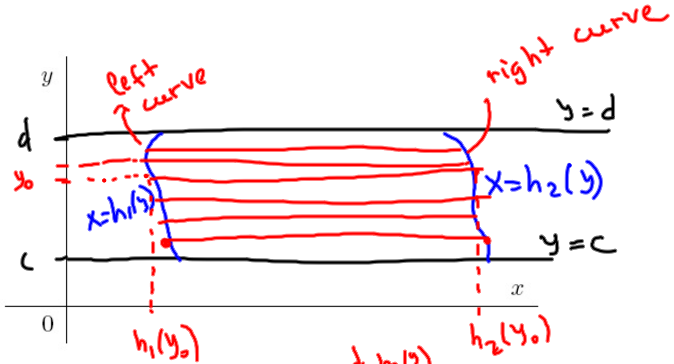
$$\iint_D f(x,y)\,\mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,\mathrm{d}y\mathrm{d}x.$$

upper curve

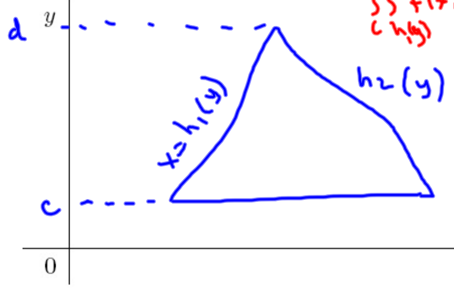
lower curve

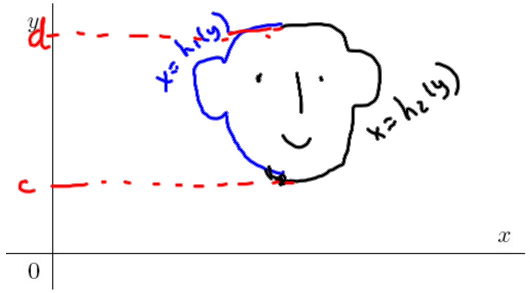
A plain region of **TYPE II**: (horizontal)

$$D = \{(x, y)|\ c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}\,.$$



$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$





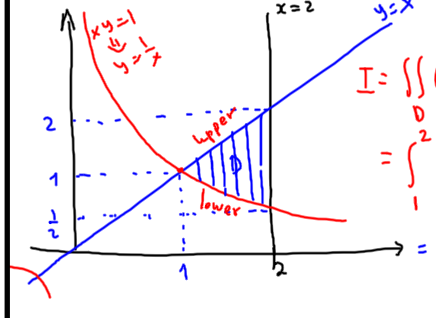
THEOREM 3. *If D is a region of type II s.t. $D = \{(x, y)|\ c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$ then*

right curve

$$\iint_D f(x, y)\, \mathrm{d}A = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, \mathrm{d}x\mathrm{d}y.$$

left curve

EXAMPLE 4. *Evaluate* $I=\iint_D (x+y)\,\mathrm{d}A$, *where* D *is the region bounded by the lines* $x=2, y=x$ *and the hyperbola* $xy=1$.



$$I=\iint_D (x+y)\,dA=$$

$$=\int_1^2\int_{\text{lower}}^{\text{upper}}(x+y)\,dy\,dx$$

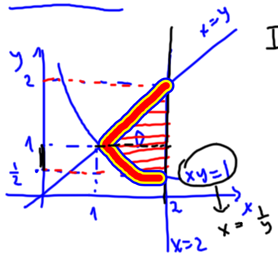
$$=\int_1^2\int_{\frac{1}{x}}^{x}(x+y)\,dy\,dx=$$

$$=\int_1^2\left(xy+\frac{y^2}{2}\right)\Bigg|_{y=\frac{1}{x}}^{x}dx=$$

$$=\int_1^2 x^2+\frac{x^2}{2}-\left(x\cdot\frac{1}{x}+\frac{1}{2x^2}\right)dx$$

$$=\int_1^2\left(\frac{3}{2}x^2-1-\frac{1}{2}x^{-2}\right)dx=\dots=\boxed{\frac{9}{4}}$$

Second Way (Use TYPE II Region)



$$I=\iint_D (x+y)\,dA=$$

$$=\int_{\frac{1}{2}}^{2}\int_{\text{left curve}}^{2}(x+y)\,dx\,dy$$

left curve $x=\begin{cases}\frac{1}{y} & \frac{1}{2}\le y\le 1\\ y & 1\le y\le 2\end{cases}$

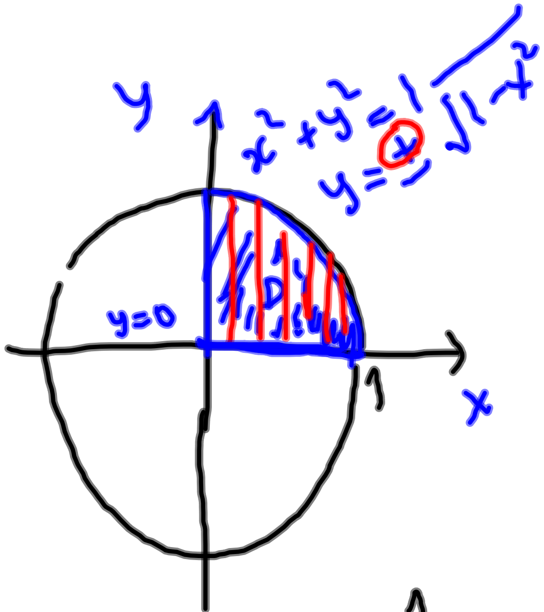
$$I=\int_{\frac{1}{2}}^{1}\int_{\frac{1}{y}}^{2}(x+y)\,dx\,dy+\int_1^2\int_y^2(x+y)\,dx\,dy=$$

$$=\int_{\frac{1}{2}}^{1}\left(\frac{x^2}{2}+xy\right)\Bigg|_{x=\frac{1}{y}}^{2}dy+\int_1^2\left(\frac{x^2}{2}+xy\right)\Bigg|_{x=y}^{2}dy=$$

$$=\dots=\frac{9}{4}$$

EXAMPLE 5. *Find the volume of the solid bounded by the cylinder* $x^2+y^2=1$ *and the planes* $x=0, y=\underbrace{z}_{\text{Lid}}, z=0$ *in the first octant.* $x \geq 0$, $y \geq 0$, $z \geq 0$

$$\boxed{V=\iint_D f(x,y)\,dA} = \iint_D y\,dA =$$



$$= \int_0^1 \int_{\text{lower}}^{\text{upper}} y\,dy\,dx = \int_0^1 \int_0^{\sqrt{1-x^2}} y\,dy\,dx =$$

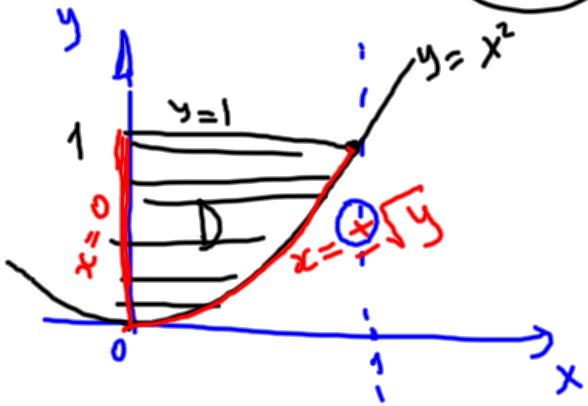
$$= \int_0^1 \frac{y^2}{2}\Big|_{y=0}^{\sqrt{1-x^2}} dx =$$

$$= \frac{1}{2}\int_0^1 (1-x^2)\,dx = \boxed{\frac{1}{3}}$$

EXAMPLE 6. *Evaluate the integral by reversing the order of integration:*

$$I = \int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\,\mathrm{d}y\mathrm{d}x.$$

Iterated integral $dy\,dx$ → double integral $\iint_D f\,dA$ → iterated integral $dx\,dy$



$$I = \iint_D x^3 \sin y^3 \, dA = \int_0^1 \int_0^{\sqrt{y}} x^3 \sin y^3 \, dx\, dy = \int_0^1 \sin y^3 \left(\int_0^{\sqrt{y}} x^3 \, dx \right) dy =$$

$$= \int_0^1 \sin y^3 \cdot \frac{x^4}{4} \Big|_0^{\sqrt{y}} dy = \frac{1}{4} \int_0^1 y^2 \sin y^3 \, dy = \left(\begin{matrix} u = y^3 \\ du = 3y^2 dy \end{matrix} \right)$$

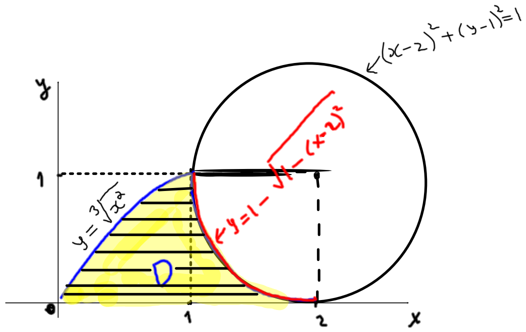
$$= \frac{1}{4} \cdot \frac{1}{3} \left(-\cos y^3 \right) \Big|_0^1 = (-) \frac{1}{12} \left(\cos 1 - \underset{\overset{\shortparallel}{1}}{\cos 0} \right) =$$

$$= \frac{1}{12} (1 - \cos 1)$$

EXAMPLE 7. *Sketch the region of integration and change the order of integration:*

$$\iint_D f(x,y)\,\mathrm{d}A = \int_0^1 \int_0^{\sqrt[3]{x^2}} f(x,y)\,\mathrm{d}y\mathrm{d}x + \int_1^2 \int_0^{1-\sqrt{1-(x-2)^2}} f(x,y)\,\mathrm{d}y\mathrm{d}x$$

Solution: First sketch the region D.



Rewrite the given curves whih are given as y=y(x) in the form x=x(y).

We have

$$y=\sqrt[3]{x^2}$$
$$\Downarrow$$
$$y^3=x^2$$
$$\Downarrow$$
$$x=\pm\sqrt{y^3}$$

Since x is nonnegative we choose $\boxed{x=\sqrt{y^3}}$

Also we have

$$y=1-\sqrt{1-(x-2)^2}$$
$$\Downarrow$$
$$(x-2)^2+(y-1)^2=1 \quad \text{(circle)}$$
$$\Downarrow$$
$$x=2\pm\sqrt{1-(y-1)^2}$$

Choose "−" because we need the left part of the circle

Finally,

$$\iint_D f(x,y)\,dA = \int_0^1 \int_{\sqrt{y^3}}^{2-\sqrt{1-(y-1)^2}} f(x,y)\,dx\,dy$$

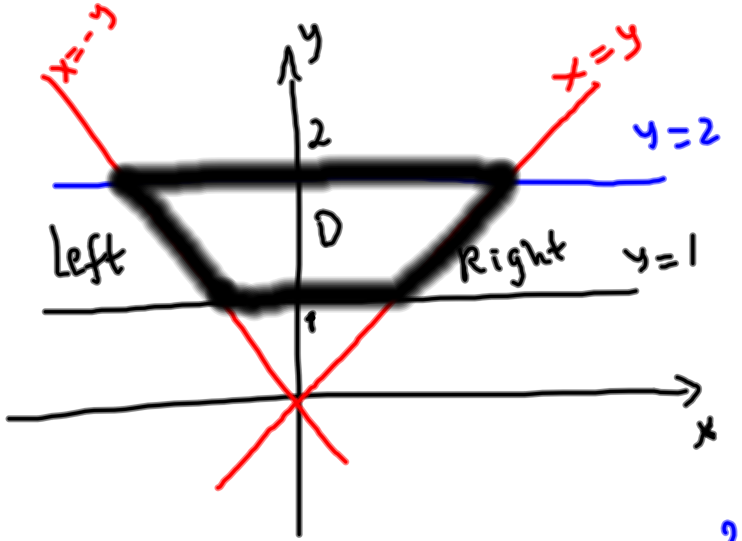
EXAMPLE 8. *Evaluate the double integral*

$$I = \iint_D e^{\frac{x}{y}}\, \mathrm{d}A$$

$$\iint e^{\frac{x}{y}}\, dx\, dy \qquad \underline{\iint \underline{e^{\frac{x}{y}}}\, dy\, dx} \not\Rightarrow$$

where D is bounded by the lines

$$y = 1, y = 2, x = -y, x = y.$$



$$I = \int_1^2 \int_{-y}^{y} e^{\frac{x}{y}}\, dx\, dy =$$

$$= \int_1^2 \left. \frac{e^{\frac{x}{y}}}{\frac{1}{y}} \right|_{x=-y}^{y} dy$$

$$= \int_1^2 y \left(e^{\frac{x}{y}} \Big|_{x=-y}^{y} \right) dy = \int_1^2 y \underbrace{\left(e^{\frac{y}{y}} - e^{\frac{-y}{y}} \right)}_{e - e^{-1}} dy =$$

$$= (e - e^{-1}) \int_1^2 y\, dy = \boxed{\frac{3}{2}\left[e - e^{-1}\right]}$$

UNION

Properties of double integrals:

- If $D = D_1 \cup D_2$, where D_1 and D_2 do not overlap except perhaps their boundaries then

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_{D_1} f(x,y)\,\mathrm{d}A + \iint_{D_2} f(x,y)\,\mathrm{d}A.$$

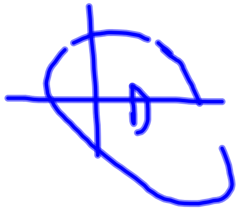
- If α and β are real numbers then

$$\iint_D (\alpha f(x,y) + \beta g(x,y))\,\mathrm{d}A = \alpha \iint_D f(x,y)\,\mathrm{d}A + \beta \iint_D g(x,y)\,\mathrm{d}A.$$

• If we integrate the constant function $f(x,y) = 1$ over D, we get area of D:

$$\iint_D dA = \iint_D 1\,\mathrm{d}A = A(D).$$

area



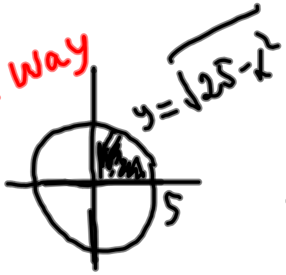
circle $r=5$

EXAMPLE 9. *If* $D = \{(x,y) |\ x^2 + y^2 \le 25\}$ *then*

I way

$$\iint_D \mathrm{d}A = \text{Area of circle} = \pi \cdot 5^2 = 25\pi$$

II way



$$A(D) = 4\int_0^5 \int_0^{\sqrt{25-x^2}} dy\, dx = 4\int_0^5 \sqrt{25-x^2}\, dx = \ldots$$

$= \ldots$ - use trigonometric substitution

$x = 5\sin\theta$