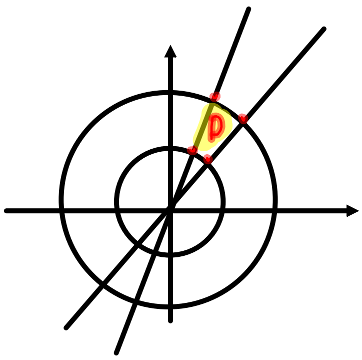
13.5: Double integrals in polar coordinates

EXAMPLE 1. *Evaluate*

$$I = \iint_D \arctan \frac{y}{x} \, \mathrm{d}A$$

where $D = \left\{(x,y) \mid 1 \le x^2 + y^2 \le 4, x \le y \le \sqrt{3}x, x \ge 0\right\}.$



$r \ge 0$ $\quad \cancel{0 \le \theta \le 2\pi}$ $\quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

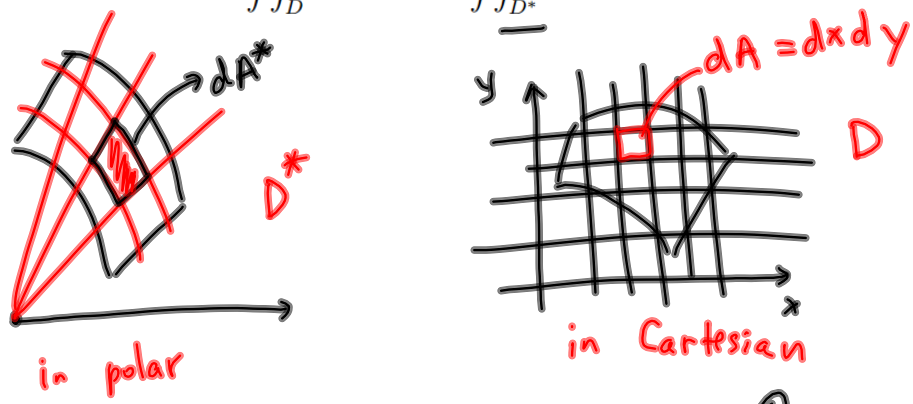
$$D^* = \left\{(r,\theta) : 1 \le r^2 \le 4, \cancel{r}\cos\theta \le \cancel{r}\sin\theta \le \sqrt{3}\cancel{r}\cos\theta\right\}$$

$$D^* = \left\{(r,\theta) : 1 \le r \le 2, 1 \le \tan\theta \le \sqrt{3}\right\}$$

$$D^* = \left\{(r,\theta) : \underbrace{1 \le r \le 2, \frac{\pi}{4} \le \theta \le \frac{\pi}{3}}_{\text{rectangle}}\right\}$$

THEOREM 2. *Change to polar coordinates in a double integral: Let f be a continuous on the region D. Denote by D^* the region representing D in the polar coordinates (r, θ). Then*

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_{D^*} f(r\cos\theta, r\sin\theta)\; r\,\mathrm{d}r\mathrm{d}\theta.$$



REMARK 3. Be careful not to forget the additional factor r on the right side of the formula.

$$\iint_D f(x,y)\,dA = \iint_{D^*} f(r\cos\theta, r\sin\theta)\,dA^*$$

$$= \iint_{D^*} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Solution of Example 1: Evaluate $I = \iint_D \arctan \frac{y}{x} \, \mathrm{d}A$

where $D = \left\{(x, y) |\ 1 \leq x^2 + y^2 \leq 4, x \leq y \leq \sqrt{3}x, x \geq 0\right\}$.

$$D^* = \left\{(r, \theta) : 1 \leq r \leq 2,\ \frac{\pi}{4} \leq \theta \leq \frac{\pi}{3}\right\}$$

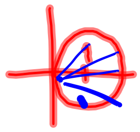
$$I = \iint_{D^*} \arctan(\tan\theta)\, dA^* = \int_{\frac{\pi}{4}}^{\pi/3}\int_1^2 \theta \, r \, dr \, d\theta$$

$$= \left(\int_{\frac{\pi}{4}}^{\pi/3} \theta \, d\theta\right)\left(\int_1^2 r \, dr\right) = \frac{\theta^2}{2}\bigg|_{\frac{\pi}{4}}^{\pi/3} \cdot \frac{r^2}{2}\bigg|_1^2$$

$$= \frac{\pi^2}{2}\left(\frac{1}{9} - \frac{1}{16}\right) \cdot \frac{4-1}{2}$$

$$= \frac{7\pi^2}{192}$$

EXAMPLE 4. *Find the volume of the solid that lies under the paraboloid* $\underbrace{z=x^2+y^2}_{\text{Lid}}$, *above the xy-plane and inside the* $\underbrace{\text{cylinder } x^2+y^2=2x}_{\text{base}}$.



base $D=\{x^2+y^2\le 2x\}$

$\Downarrow \quad x=r\cos\theta\ge 0$

$D^*=\{r^2\le 2r\cos\theta,\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\}$

$D^*=\{0\le r\le 2\cos\theta,\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\}$

$$V=\iint_D \underbrace{(x^2+y^2)}_{r^2}\,dA$$

$$V=\iint_{D^*} r^2\,dA^*=\int_{-\frac{\pi}{2}}^{\pi/2}\int_0^{2\cos\theta}\underbrace{r^2\,r}_{r^3}\,dr\,d\theta=\int_{-\frac{\pi}{2}}^{\pi/2}\frac{r^4}{4}\Big|_0^{2\cos\theta}d\theta$$

$\cos^2\theta=\frac{1+\cos2\theta}{2}$

$$=\frac{1}{4}\int_{-\frac{\pi}{2}}^{\pi/2}2^4\cos^4\theta\,d\theta=4\int_{-\frac{\pi}{2}}^{\pi/2}\frac{(1+\cos2\theta)^2}{2^2}d\theta$$

$$=\int_{-\pi/2}^{\pi/2}\underbrace{1+2\cos2\theta}+\cos^2 2\theta\,d\theta$$

$$=\theta-\sin2\theta\Big|_{-\frac{\pi}{2}}^{\pi/2}+\int_{-\frac{\pi}{2}}^{\pi/2}\frac{1+\cos4\theta}{2}d\theta=\dots=\boxed{\frac{3\pi}{2}}$$

EXAMPLE 5. *Find the area of the region inside the circle* $\underbrace{r = 4\sin\theta}$ *and outside the circle* $\underbrace{r = 2}$.

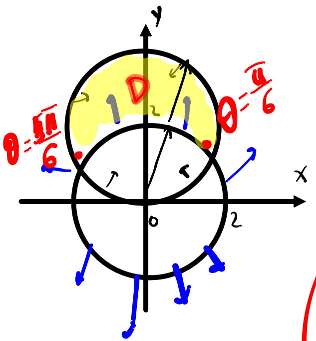
$$\text{Area}(D) = \iint_D dA$$

$r^2 = 4r\sin\theta$

$x^2+y^2 = 4y$

$x^2 + (y-2)^2 = 4$

$x^2+y^2=4$



$$D^* = \left\{(r,\theta) : 2 \le r \le 4\sin\theta,\ \frac{\pi}{6} \le \theta \le \frac{5\pi}{6}\right\}$$

To determine bounds for the angle we need to find the angles corresponding to intersection points.

$$\begin{cases} r = 4\sin\theta \\ r = 2 \end{cases} \Rightarrow 4\sin\theta = 2 \Rightarrow \sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\text{Area}(D) = \iint_D dA = \iint_{D^*} r\,dr\,d\theta = \int_{\pi/6}^{5\pi/6}\int_2^{4\sin\theta} r\,dr\,d\theta$$

$$= \int_{\pi/6}^{5\pi/6} \frac{r^2}{2}\Big|_{r=2}^{4\sin\theta} d\theta = \int_{\pi/6}^{5\pi/6} (8\sin^2\theta - 2)\,d\theta$$

$$= \int_{\pi/6}^{5\pi/6} 4(1-\cos 2\theta)\,d\theta - 2\left(\frac{5\pi}{6} - \frac{\pi}{6}\right) = \ldots$$