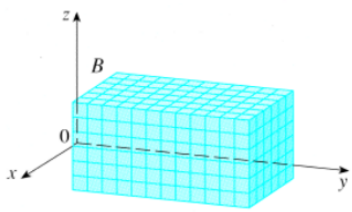
13.8: Triple Integrals $\rho = \text{const} \Rightarrow m = \rho V$

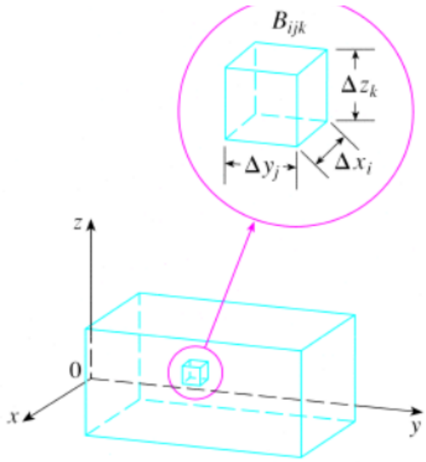
Mass problem: *Given a solid object, that occupies the region B in $\mathbb{R}^3$, with density $\rho(x, y, z)$. Find the mass of the object.*

Solution: Let B be a rectangular box:



$$B = \{(x, y, z) | a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}$$
$$= [a, b] \times [c, d] \times [r, s]$$

Partition in sub-boxes:



$$m_{ijk} = \rho(x_i^*, y_j^*, z_k^*)\Delta V_{ijk}$$

$$\|P\| = \max \sqrt{\Delta x_i^2 + \Delta y_j^2 + \Delta z_k^2}$$

$$m = \lim_{\|P\| \to 0} \sum_i \sum_j \sum_k \rho(x_i^*, y_j^*, z_k^*)\Delta V_{ijk}$$

$$m = \iiint_B \rho(x, y, z)\, \mathrm{d}V$$

FUBINI's THEOREM: *If f is continuous on the rectangular box $B = [a,b] \times [c,d] \times [r,s]$ then*

iterated integral

$$\iiint_B f(x,y,z)\,\mathrm{d}V = \underbrace{\int_r^s \int_c^d \int_a^b f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\mathrm{d}z}$$

$dx\,dz\,dy$

and there are 5 *other possible orders in which we can integrate.*

$dy\,dz\,dx$

⋮

EXAMPLE 1. *Let* $B = [0,1] \times [-1,3] \times [0,3]$. *Evaluate*

$x \quad y \quad z$

$$I = \iiint_B xye^{yz}\,\mathrm{d}V$$

$$I = \int_{-1}^{3}\int_{0}^{3}\int_{0}^{1} \underline{x}\, y e^{yz}\, \underline{dx}\, dz\, dy$$

$$= \left(\int_0^1 x\,dx\right) \cdot \int_{-1}^{3} y \left(\int_0^3 e^{yz}\,dz\right) dy$$

$$= \frac{1}{2}\int_{-1}^{3} \cancel{y}\, \frac{e^{yz}}{\cancel{y}}\Bigg|_{z=0}^{3} dy = \frac{1}{2}\int_{-1}^{3}\left(e^{3y}-1\right)dy$$

$$= \ldots$$

FACT: The volume of the solid E is given by the integral,

$$V = \iiint_E dV.$$

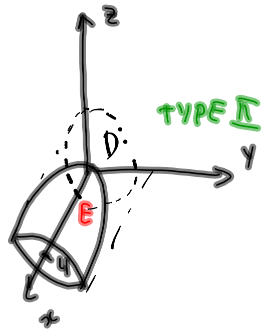
FACT: The mass of the solid E with variable density $\rho(x, y, z)$ is given by the integral,

$$m = \iiint_E \rho(x,y,z)\, dV.$$

EXAMPLE 2. *Find the mass of the solid bounded by* $\underbrace{x = y^2 + z^2}_{\text{parboloid}}$ *and the plane* $x = 4$ *if the density function is* $\rho(x, y, z) = \sqrt{y^2 + z^2}$.

$$m = \iiint_E \sqrt{y^2+z^2}\, dV$$

$$= \iint_D \left(\int_{y^2+z^2}^{4} \sqrt{y^2+z^2}\, dx \right) dA$$



The boundary of D is the line of intersection:

$$\left.\begin{aligned} x &= y^2+z^2 \\ x &= 4 \end{aligned}\right\} \Rightarrow y^2+z^2 = 4$$

$$D = \{(y,z) : y^2+z^2 \le 4\}$$

$$m = \iint_D \sqrt{y^2+z^2} \left(\int_{y^2+z^2}^{4} dx \right) dA = \iint_D \sqrt{y^2+z^2}\,\left(4-(y^2+z^2)\right) dA$$

Use polar coordinates: $\left.\begin{aligned} y &= r\cos\theta \\ z &= r\sin\theta \end{aligned}\right\} \Rightarrow y^2+z^2 = r^2$

$$D^* = \{(r,\theta) : 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$$

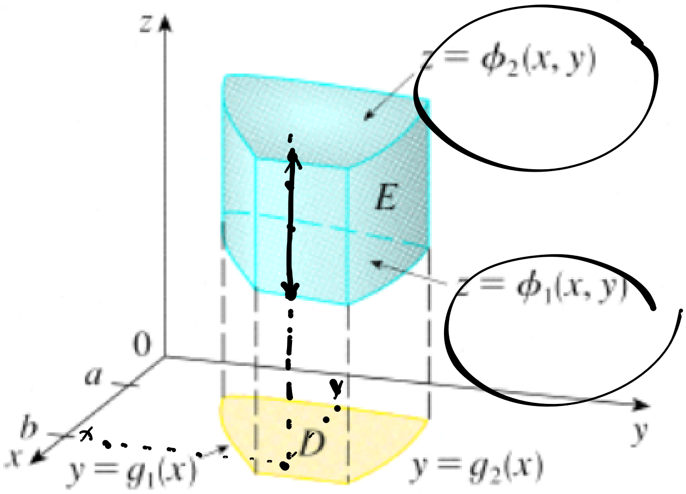
$$m = \iint_{D^*} r(4-r^2)\, dA^* = \int_0^{2\pi}\int_0^2 (4r - r^3)\, r\, dr\, d\theta$$

$$= 2\pi \int_0^2 (4r^2 - r^4)\, dr = \ldots = \frac{128\pi}{15}$$

A solid region of **TYPE I**:

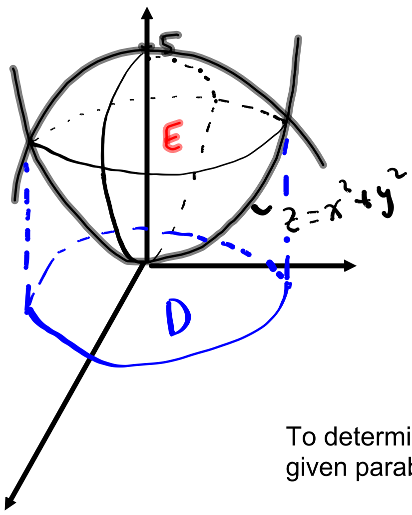
$E = \{(x, y, z) | (x, y) \in D, \phi_1(x, y) \leq z \leq \phi_2(x, y)\}$
where D is the projection of E onto the xy-plane.

A type 1 solid region



$$\iiint f(x, y, z)\,\mathrm{d}V = \iint \left[\int^{\phi_2(x,y)} f(x, y, z)\,\mathrm{d}z \right] \mathrm{d}A$$

EXAMPLE 3. *Use a triple integral to find the volume of the solid bounded by the surfaces* $\underbrace{z = x^2 + y^2}_{\leftarrow \text{paraboloid}}$ *and* $z = 5 - 4x^2 - 4y^2$.



$$V = \iiint_E dV = \iint_D \left(\int_{x^2+y^2}^{5-4x^2-4y^2} dz \right) dA$$

To determine D note that its boundary coincides with the line of intersection of two given paraboloids:

$$\left.\begin{array}{l} z = x^2+y^2 \\ z = 5-4(x^2+y^2) \end{array}\right\} \Rightarrow x^2+y^2 = 5-4(x^2+y^2)$$

$$5(x^2+y^2) = 5$$

$$x^2+y^2 = 1$$

Thus $D = \{ (x,y) : x^2+y^2 \le 1 \}$

$$V = \iint_D z \Big|_{z=x^2+y^2}^{z=5-4(x^2+y^2)} dA = \iint_D 5-4(x^2+y^2)-(x^2+y^2)\, dA$$

$$= \iint_D 5(1-(x^2+y^2))\, dA = \iint_{D^*} 5(1-r^2)\, dA^*$$

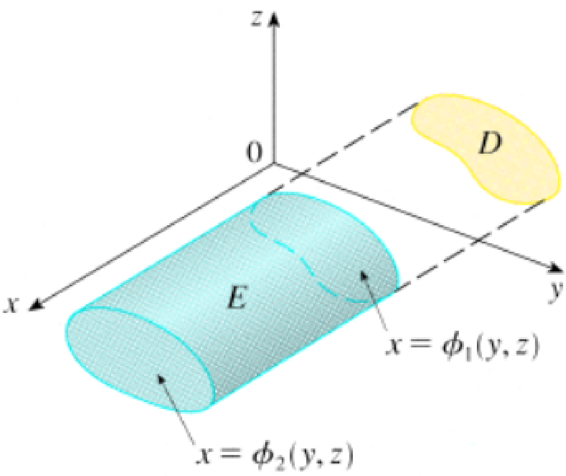
where $D^* = \{ (r,\theta) : 0 \le r \le 1, \ 0 \le \theta \le 2\pi \}$

$$V = 5\int_0^{2\pi}\int_0^1 (1-r^2)\, r\, dr\, d\theta = 5 \cdot 2\pi \int_0^1 r - r^3 \, dr$$

$$= 10\pi \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{5\pi}{2}$$

A solid region of **TYPE II**:

$E = \{(x, y, z) | (y, z) \in D, \phi_1(y, z) \leq x \leq \phi_2(y, z)\}$
where D is the projection of E onto the yz-plane.

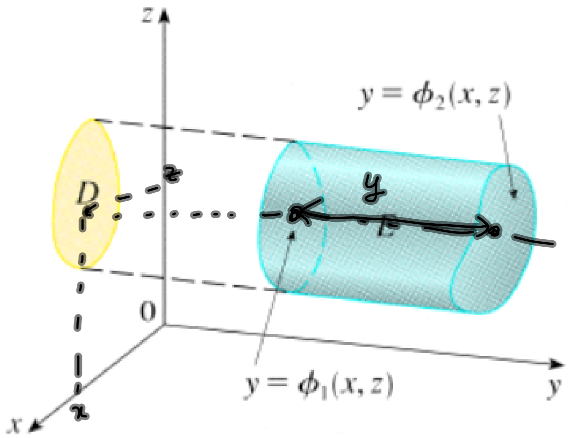


A type 2 region

$$\iiint_E f(x, y, z)\, \mathrm{d}V = \iint_D \left[\int_{\varphi_1(y,z)}^{\varphi_2(y,z)} f(x, y, z)\, \mathrm{d}x\right] \mathrm{d}A$$

A solid region of **TYPE III**:

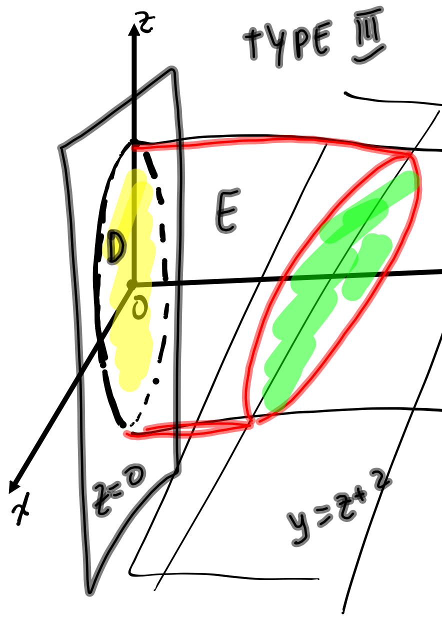
$E = \{(x,y,z)|(x,z) \in D, \phi_1(x,z) \leq y \leq \phi_2(x,z)\}$ where D is the projection of E onto the xz-plane.



A type 3 region

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \iint_D \left[\int_{\varphi_1(x,z)}^{\varphi_2(x,y)} f(x,y,z)\,dy \right] dA$$

EXAMPLE 4. *Use a triple integral to find the* volume *of the solid bounded by the elliptic cylinder* $4x^2 + z^2 = 4$ *and the planes* $\underbrace{y = 0}_{xz\text{-plane}}$ *and* $y = z + 2$.

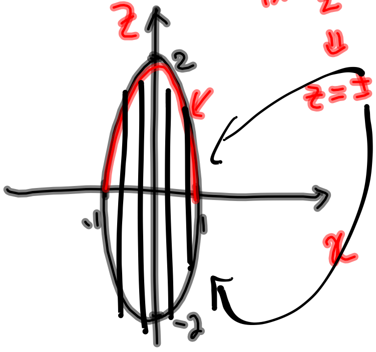


$$V = \iiint_E dV = \iint_D \left[\int_0^{z+2} dy \right] dA$$

where

$$D = \{(x,z) : 4x^2 + z^2 \le 4\}$$

$$V = \iint_D (z+2)\, dA = \int_{-1}^{1} \int_{-\sqrt{4-4x^2}}^{\sqrt{4-4x^2}} (z+2)\, dz\, dx$$



$$= \int_{-1}^{1} \left(\frac{z^2}{2} + 2z \right) \Bigg|_{z=-2\sqrt{1-x^2}}^{z=2\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1} 0 \quad + 2 \cdot 4\sqrt{1-x^2}\, dx$$

$$= 8 \underbrace{\int_{-1}^{1} \sqrt{1-x^2}\, dx}_{\text{area of half circle with radius}} = 8 \cdot \frac{\pi}{2} = 4\pi.$$