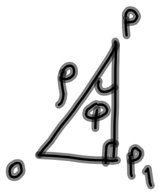
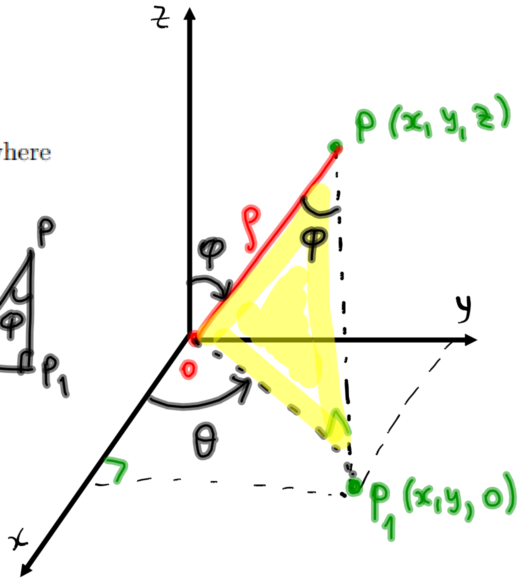
13.9-13.10: Part II

Triple integrals in spherical coordinates

- **Spherical coordinates** of P is the ordered triple (ρ, θ, ϕ) where $|OP| = \rho$, $\rho \geq 0$, $0 \leq \theta \leq 2\pi$, $\boxed{0 \leq \phi \leq \pi.}$





$|OP_1| = \underbrace{\rho \sin\varphi}$

$x = |OP_1| \cos\theta = \rho \sin\varphi \cos\theta$

$y = |OP_1| \sin\theta = \rho \sin\varphi \sin\theta$

$z = |PP_1| = \rho \cos\varphi$

We have

$$\boxed{\begin{aligned} x &= \rho \sin\varphi \cos\theta \\ y &= \rho \sin\varphi \sin\theta \\ z &= \rho \cos\varphi \end{aligned}}$$

$(x, y, z) \rightarrow (\rho, \theta, \varphi)$

REMARK 1. The spherical coordinates

$$\begin{aligned}
x &= \rho \sin\phi \cos\theta \\
y &= \rho \sin\phi \sin\theta \\
z &= \rho \cos\phi \\
\rho &\geq 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi
\end{aligned}$$

are especially useful in problems where there is *symmetry about the origin.*

Note that

$$x^2 + y^2 + z^2 = \rho^2 \sin^2\varphi \cos^2\theta + \rho^2 \sin^2\varphi \sin^2\theta + \rho^2 \cos^2\varphi$$

$$= \rho^2 \{ \underbrace{\sin^2\varphi \cos^2\theta + \sin^2\varphi \sin^2\theta} + \cos^2\varphi \}$$

$$= \rho^2 \{ \sin^2\varphi (\underbrace{\cos^2\theta + \sin^2\theta}_{=1}) + \cos^2\varphi \}$$

$$= \rho^2 (\underbrace{\sin^2\varphi + \cos^2\varphi}_{=1}) = \rho^2$$

$$\boxed{x^2 + y^2 + z^2 = \rho^2}$$

EXAMPLE 2. *Find equation in spherical coordinates for the following surfaces.*

(a) $\underbrace{x^2 + y^2 + z^2}_{} = 16$

$$\rho^2 = 16 \Rightarrow \boxed{\rho = 4}$$

(b) $z = \sqrt{x^2 + y^2}$

$$\rho \cos\varphi = \sqrt{\rho^2(\sin^2\varphi\cos^2\theta + \sin^2\varphi\sin^2\theta)}$$

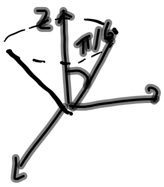
$$\rho \cos\varphi = \rho\sqrt{\sin^2\varphi\underbrace{(\cos^2\theta + \sin^2\theta)}_{=1}}$$

$$\rho\cos\varphi = \rho\sin\varphi \Rightarrow \tan\varphi = 1 \Rightarrow \boxed{\varphi = \frac{\pi}{4}}$$

(c) $z = \sqrt{3x^2 + 3y^2}$ $\Rightarrow$ $z = \sqrt{3}\sqrt{x^2+y^2}$ see part (b)

cone

$$\rho \cos\varphi = \sqrt{3}\,\rho \sin\varphi$$



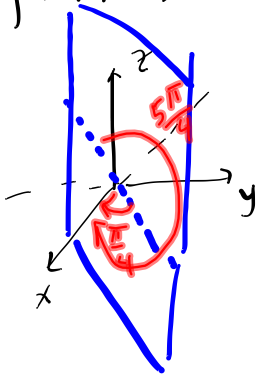
$$\tan\varphi = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \Rightarrow \boxed{\varphi = \frac{\pi}{6}}$$

(d) $x = y$

$$\rho \sin\varphi \cos\theta = \rho \sin\varphi \sin\theta \Rightarrow \cos\theta = \sin\theta \Rightarrow \tan\theta = 1$$

$$\Downarrow$$

$$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$$

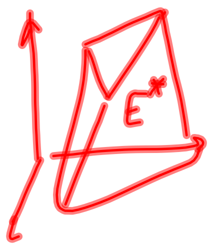


$$\boxed{\theta = \frac{\pi}{4} \ \cup \ \theta = \frac{5\pi}{4}}$$

•**Triple integrals in spherical coordinates**

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$
$$r \geq 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi$$

spherical coordinates



$$(x, y, z) \quad \rightarrow \quad \boxed{(r, \theta, z)} \quad \rightarrow \quad \boxed{(\rho, \theta, \phi)}$$

$$x = r\cos\theta \qquad z = \rho\cos\phi$$
$$y = r\sin\theta \qquad r = \rho\sin\phi$$
$$z = z \qquad \underbrace{\theta = \theta}_{\text{cylindrical coord.}}$$

$$dV = dx\,dy\,dz =$$

$$= r\,dr\,d\theta\,dz = r\,\rho\,d\rho\,d\theta\,d\varphi$$

$$= \rho\sin\varphi\,\rho\,d\rho\,d\theta\,d\varphi$$

$$\boxed{dV = dx\,dy\,dz = \rho^2 \sin\varphi\,d\rho\,d\theta\,d\varphi}$$

$$0 \leq \varphi \leq \pi \Rightarrow \sin\varphi \geq 0 \Rightarrow dV \geq 0$$

THEOREM 3. *Let $f(x, y, z)$ be a continuous function over a solid $E \subset \mathbb{R}^3$. Let E^* be its image in spherical coordinates. Then*

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \iiint_{E^*} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\underbrace{\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi}.$$

EXAMPLE 4. *Evaluate* $I = \iiint_E e^{\sqrt{(x^2+y^2+z^2)^3}}\,\mathrm{d}V$ *where* $E = \{(x,y,z) : 9 \le x^2+y^2+z^2 \le 16\}$

$\boxed{x^2+y^2+z^2=\rho^2}$

$$E^* = \{(\rho, \theta, \varphi) : 3 \le \rho \le 4,\ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi\}$$

$$I = \iiint_{E^*} e^{\sqrt{(\rho^2)^3}} \rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_3^4 e^{\rho^3}\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin\varphi\, d\varphi\right)\left(\int_3^4 e^{\rho^3}\rho^2\, d\rho\right)$$

$$= 2\pi\,(-\cos\varphi)\Big|_0^{\pi}\ \frac{1}{3}e^{\rho^3}\Big|_3^4$$

$$= 2\pi\cdot 2\cdot\frac{1}{3}\left(e^{64}-e^{27}\right) = \frac{4\pi}{3}\left(e^{64}-e^{27}\right)$$

EXAMPLE 5. *Write the integral* $\iiint_E f(x,y,z)\,\mathrm{d}V$ *in spherical coordinates where*

(a) $E = \{(x,y,z) : x^2+y^2+z^2 \le 1, y \ge 0, z \ge 0\}$.

$$E^* = \{(\rho,\theta,\varphi) : 0 \le \rho \le 1,\ 0 \le \theta \le \pi,\ 0 \le \varphi \le \tfrac{\pi}{2}\}$$

$$\iiint_E f(x,y,z)\,dV = \int_0^{\pi/2}\int_0^{\pi}\int_0^{1} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

(b) E *is the icecream cone-shaped solid, which is cut from the sphere of radius 5 by the cone* $\phi = \pi/6$.

$$E^* = \{(\rho,\theta,\varphi) : 0 \le \rho \le 5,\ 0 \le \varphi \le \tfrac{\pi}{6},\ 0 \le \theta \le 2\pi\}$$

$$I = \iiint_E f(x,y,z)\,dV = \int_0^{2\pi}\int_0^{\pi/6}\int_0^{5} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

EXAMPLE 6. *Evaluate the integral by changing to spherical coordinates:*

$$I = \int_0^3 \int_0^{\sqrt{9-y^2}} \left[\int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)\,dz\right] dx\,dy.$$

iterated integral

projection of E onto the xy-plane

$$\iint_D \ldots dA$$

$$= \iiint_E (x^2+y^2+z^2)\,dV$$

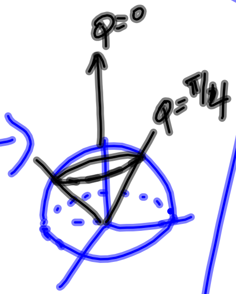
Solid is bounded by two surfaces

$\underbrace{z=\sqrt{x^2+y^2}}$ $\varphi = \frac{\pi}{4}$ and $z=\sqrt{18-x^2-y^2} \Rightarrow \rho = 3\sqrt{2}$

upper half sphere with radius $\sqrt{18}=3\sqrt{2}$ $\quad 0 \le \varphi \le \frac{\pi}{2}$

And projection of E onto the xy-plane is the region

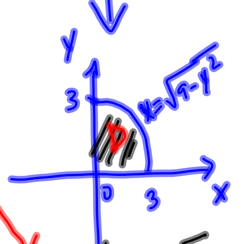
$D = \{0 \le y \le 3,\ 0 \le x \le \sqrt{9-y^2}\}$



$\boxed{0 \le \varphi \le \frac{\pi}{4}}$ $\quad \boxed{0 \le \rho \le 3\sqrt{2}}$

To determine bounds for θ look at D (projection)

$\boxed{0 \le \theta \le \frac{\pi}{2}}$



$$I = \int_0^{\pi/4}\int_0^{\pi/2}\int_0^{3\sqrt{2}} \underbrace{\rho^2\,\rho^2}_{\rho^4} \sin\varphi \,d\rho\,d\theta\,d\varphi$$