14.1: Vector Fields

A vector function

$$\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

is an example of a function whose domain is a set of real numbers and whose range is a set of vectors in $\mathbb{R}^3$:

$$\mathbf{r}(t) : \mathbb{R} \to \mathbb{R}^3.$$

Consider a type of functions (**vector fields**) whose domain is $\mathbb{R}^2$ (or $\mathbb{R}^3$) and whose range is a set of vectors in $\mathbb{R}^2$ (or $\mathbb{R}^3$):

$$\vec{r}(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} \qquad \mathbb{R} \to \mathbb{R}^2$$

surface in $\mathbb{R}^3$ $\left\{ \vec{r}(u,v) = x(u,v)\hat{i} + y(u,v)j + z(u,v)\hat{k} \qquad \mathbb{R} \to \mathbb{R}^3 \right.$

Vector field over $\mathbb{R}^2$.

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} = \langle P(x, y), Q(x, y)\rangle$$

Vector field over $\mathbb{R}^3$:

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k} = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$$

EXAMPLE 1. *Describe the vector field* $\mathbf{F}(x,y) = -y\mathbf{i} + x\mathbf{j}$ *by sketching.*

$\vec{F}(0,0) = \vec{0}$

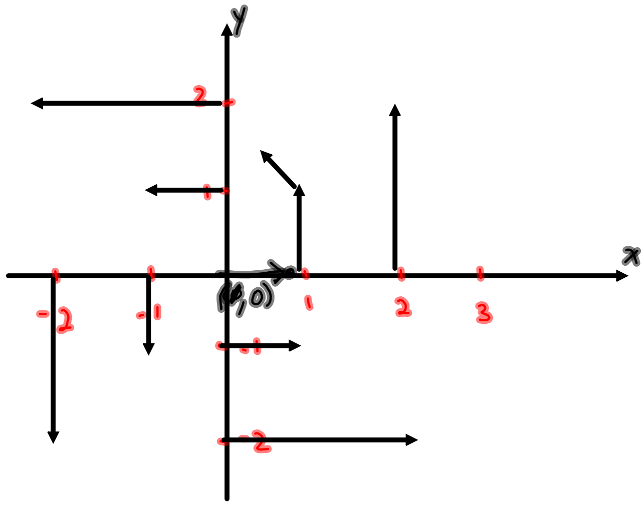
$\vec{F}(1,0) = \langle 0, 1\rangle$

$\vec{F}(0,1) = \langle -1, 0\rangle$

$\vec{F}(-1,0) = \langle 0, -1\rangle$

$\vec{F}(0,-1) = \langle 1, 0\rangle$

$\vec{F}(1,1) = \langle -1, 1\rangle$



$$|\vec{F}(x,y)| = |\langle -y, x\rangle| = \sqrt{x^2+y^2} = |\langle x, y\rangle|$$

distance from the origin

$$\vec{F}(x,y)\cdot\langle x,y\rangle = \langle -y,x\rangle\cdot\langle x,y\rangle = -yx+xy=0$$

$\vec{F}(x,y)$ at each point is perpendicular to the position vector of that point and its magnitude coincides with the position vector of that point.

Function $u = f(x, y, z)$ is also called a **scalar field**. Its gradient is also called **gradient vector field**:

$$\mathbf{F}(x, y, z) = \nabla f(x, y, z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle$$

EXAMPLE 2. *Find the gradient vector field of* $f(x, y, z) = xyz$.

$$\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle = \boxed{\langle yz, xz, xy \rangle}$$