14.2: Line Integrals

Line integrals on plane: Let C be a plane curve with parametric equations:

$$x = x(t), y = y(t), \quad a \le t \le b,$$

parameter domain

or we can write the parametrization of the curve as a vector function:

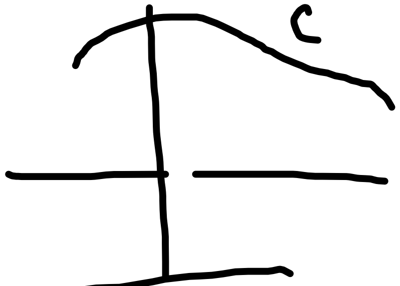
$$C: \quad \mathrm{r}(t) = \langle x(t), y(t) \rangle\,, \quad a \le t \le b.$$

DEFINITION 1. *The line integral of $f(x, y)$ with respect to arc length, or the* **line integral of f along C** *is*

$$\int_C f(x, y)\,\mathrm{d}s$$

Recall that the *arc length* of a curve given by parametric equations $x = x(t), y = y(t), \quad a \leq t \leq b$ can be found as

$$L = \int_a^b \mathrm{d}s,$$



where

$$\mathrm{d}s = \sqrt{(x'(t))^2 + (y'(t))^2}\,\mathrm{d}t.$$

The line integral is then

$$\int_C f(x,y)\,\mathrm{d}s = \int_a^b f(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

If we use the vector form of the parametrization we can simplify the notation up noticing that

$$\mathrm{r}'(t) = \langle x'(t), y'(t)\rangle$$

and then

$$\mathrm{d}s = \underbrace{\sqrt{(x'(t))^2 + (y'(t))^2}}_{|\vec{r}'(t)|}\,\mathrm{d}t = |\vec{r}'(t)|\,dt$$

Using this notation the line integral becomes,

$$\int_C f(x, y)\,\mathrm{d}s = \int_a^b f(x(t), y(t))|\mathbf{r}'(t)|\,\mathrm{d}t.$$

REMARK 2. The value of the line integral does not depend on the parametrization of the curve, provided that *the curve is traversed exactly once as t increases from a to b.*

Let us emphasize that $\boxed{\mathrm{d}s = |r'(t)|\,\mathrm{d}t} = \sqrt{[x'(t)]^2 + [y'(t)]^2}\,\mathrm{d}t.$

EXAMPLE 3. *Evaluate the line integral* $\int_C y\,\mathrm{d}s$, *where* $C: x = t^3, y = t^2, 0 \le t \le 1$.

$$\int_C y\,ds = \int_0^1 t^2 \sqrt{(3t^2)^2 + (2t)^2}\,dt =$$

$$= \int_0^1 t^2 \sqrt{9t^4 + 4t^2}\,dt =$$

$$= \int_0^1 t^3 \sqrt{9t^2 + 4}\,dt =$$

use
u-subst.
$u = 9t^2 + 4$

. . . . .

Line integrals in space: Let C be a space curve with parametric equations:

$$C: \quad x = x(t), y = y(t), z = z(t), \quad a \le t \le b,$$

or

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a \le t \le b.$$

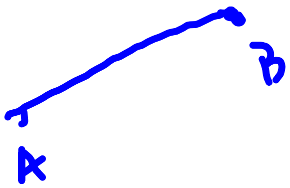
The **line integral of** f **along** C is

$$\int_C f(x,y,z)\,\mathrm{d}s = \int_a^b f(x(t),y(t),z(t))|r'(t)|\,\mathrm{d}t. = \int_a^b f(\vec{r}(t)) \cdot |r'(t)| dt$$

Here

$$\boxed{\mathrm{d}s = |r'(t)|\,\mathrm{d}t} = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,\mathrm{d}t.$$

EXAMPLE 4. *Evaluate the line integral* $\int_C (x+y+z)\,\mathrm{d}s$, *where* C *is the line segment joining* A $(-1,1,2)$ *and* B $(2,3,1)$.



Param. C

$$\vec{AB} = \langle 2,3,1\rangle - \langle -1,1,2\rangle = \langle 3,2,-1\rangle$$

$$\begin{cases} x = -1+3t \\ y = 1+2t \\ z = 2-t \end{cases}$$

$0 \le t \le 1$

$$\vec{r}(t) = \langle -1+3t,\ 1+2t,\ 2-t\rangle$$

$$\begin{cases} \vec{r}(0) = A \\ \vec{r}(1) = B \end{cases}$$

$$\vec{r}\,'(t) = \langle 3,2,-1\rangle$$

$$|\vec{r}\,'(t)| = \sqrt{3^2+2^2+(-1)^2} = \sqrt{14}$$

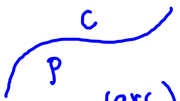
$$\int_C (x+y+z)\,\overset{= |\vec{r}\,'(t)|\,dt}{ds} = \int_0^1 (-1+3t+1+2t+2-t)\sqrt{14}\,dt =$$

$$= 4\sqrt{14}$$

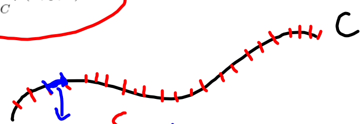
Physical interpretation of a line integral: Let $\rho(x, y, z)$ represents the linear density at a point (x, y, z) of a thin wire shaped like a curve C. Then the **mass** m of the wire is:

$$m = \int_C \rho(x, y, z)\,\mathrm{d}s.$$

?



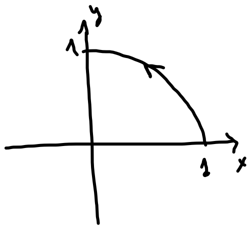
$$m = \rho \cdot \begin{pmatrix} \text{arc} \\ \text{length} \end{pmatrix} = \rho \int_C ds$$



$$\sum_i m_i = \sum_i \rho_i \, ds$$

$$\int_C \rho \, ds$$

EXAMPLE 5. *A thin wire with the linear density* $\rho(x, y) = x^2 + 2y^2$ *takes the shape of the curve* C *which consists of the arc of the circle* $x^2 + y^2 = 1$ *from* $(1, 0)$ *to* $(0, 1)$*. Find the mass of the wire.*



$$m = \int_C \rho(x,y)\, ds =$$

$$= \int_C (x^2 + 2y^2)\, ds$$

Param. C

$\vec{r}(t) = \langle \cos t, \sin t \rangle \Rightarrow \vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

$0 \le t \le \frac{\pi}{2}$ $\qquad |\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t} = 1$

$$m = \int_0^{\pi/2} (\cos^2 t + 2\sin^2 t) \cdot 1 \cdot dt =$$

$\cos^2 t + \sin^2 t + \sin^2 t$

$$= \int_0^{\pi/2} (1 + \sin^2 t)\, dt = \frac{\pi}{2} + \int_0^{\pi/2} \frac{1 - \cos 2t}{2}\, dt$$

$$= \dots = \frac{3\pi}{4}$$

Line integrals with respect to x, y, and z. Let C be a space curve with parametric equations:

$$x = x(t), y = y(t), z = z(t), \quad a \le t \le b,$$

$dx = x'(t)\,dt$

The **line integral of f with respect to** x is,

$$\int_C f(x,y,z)\,\mathrm{d}x = \int_a^b f(x(t), y(t), z(t))x'(t)\,\mathrm{d}t.$$

The **line integral of f with respect to** y is,

$$\int_C f(x,y,z)\,\mathrm{d}y = \int_a^b f(x(t), y(t), z(t))\underbrace{y'(t)\,\mathrm{d}t}_{dy}.$$

The **line integral of f with respect to** z is,

$$\int_C f(x,y,z)\,\mathrm{d}z = \int_a^b f(x(t), y(t), z(t))\,z'(t)\,dt$$

These two integral often appear together by the following notation:

$$\int_C P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$$

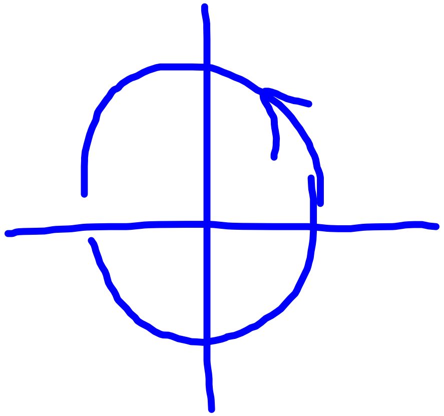
or

$$\int_C P\,\mathrm{d}x + Q\,\mathrm{d}y.$$

EXAMPLE 6. *Compute*

$$I = \int_C -\frac{y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y,$$

where C is the circle $x^2+y^2=1$ oriented in the counterclockwise direction.



$x = \cos t, \quad y = \sin t$

$dx = -\sin t\, dt, \quad y' = \cos t\, dt$

$0 \le t \le 2\pi$

$$I = \int_0^{2\pi} \left(- \frac{\sin t \cdot (-\sin t)}{\underbrace{\cos^2 t + \sin^2 t}_{=1}} + \frac{\cos t \cdot \cos t}{\underbrace{\cos^2 t + \sin^2 t}_{=1}} \right) dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt = \int_0^{2\pi} dt = 2\pi$$

Line integrals of vector fields.

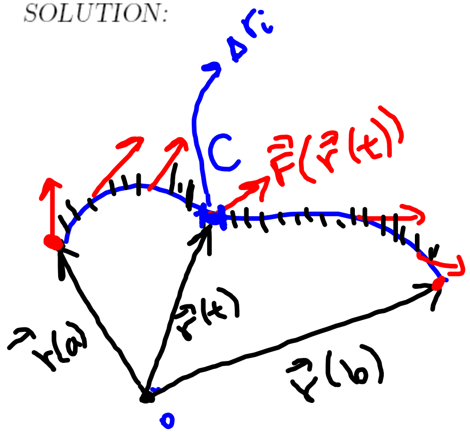
PROBLEM: Given a continuous force field,

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k},$$

such as a gravitational field. Find the work done by the force $\mathbf{F}$ in moving a particle along a curve

$$C: \quad \mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \qquad a \le t \le b.$$

SOLUTION:



If $\vec{F}$ is constant

then $W = \vec{F} \cdot d$

dot product

displacement

$$W_i = \vec{F}(\vec{r}(t_i)) \cdot \Delta\vec{r}_i$$

$$W = \lim \sum_i W_i =$$

$$= \lim \sum_i \vec{F}(\vec{r}(t_i)) \cdot \Delta\vec{r}_i$$

$$W = \int_C \vec{F} \cdot d\vec{r}$$

$$\boxed{d\vec{r} = \vec{r}\,'(t)\,dt}$$ memo

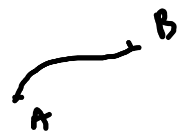
DEFINITION 7. *Let* $\mathbf{F}$ *be a continuous vector field defined on a curve* C *given by a vector function* $\mathbf{r}(t)$, $a \leq t \leq b$. *Then the* **line integral of F along** C *is*

dot product

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}(t) = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \underbrace{\mathbf{r}'(t)\,\mathrm{d}t}_{d\vec{r}(t)}.$$

REMARK 8. Note that this integral depends on the curve orientation:

$$\int_{-C} \mathbf{F} \cdot \mathrm{d}\mathbf{r}(t) = -\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}(t)$$



EXAMPLE 9. *Find the work done by the force field* $\mathbf{F}(x, y, z) = \langle xy, yz, xz \rangle$ *in moving a particle along the curve* $C : \mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, $0 \leq t \leq 1$.

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_0^1 \langle t \cdot t^2, t^2 \cdot t^3, t \cdot t^3 \rangle \cdot \langle 1, 2t, 3t^2 \rangle dt$$

$$d\vec{r} = \vec{r}\,'(t)\,dt = \langle 1, 2t, 3t^2 \rangle dt$$

$$W = \int_0^1 \left(t^3 \cdot 1 + t^5 \cdot 2t + t^4 \cdot 3t^2 \right) dt =$$

$$= \ldots = \boxed{\frac{27}{28}}$$

Relationship between line integrals of vector fields and line integrals with respect to x, y, and z. $\vec{F} = \langle P, Q, R \rangle$, $d\vec{r} = \langle dx, dy, dz \rangle$

$$\int_C \mathbf{F} \cdot d\mathbf{r}(t) = \int_C \langle P, Q, R \rangle \cdot \langle dx, dy, dz \rangle =$$

dot product

$$= \int_C P\,dx + Q\,dy + R\,dz =$$

$$= \int_C P\,dx + \int_C Q\,dy + \int_C R\,dz$$

$$= \int_a^b P(\vec{r}(t))\,x'(t)\,dt + \int_a^b Q(\vec{r}(t))\,y'(t)\,dt + \int_a^b R(\vec{r}(t))\,z'(t)\,dt$$

For example,

Ex A $\int_C x\,dx + y^2 \sin xy\,dy$

Ex. B $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = \langle x, y^2 \sin xy \rangle$

The integrals in Ex. A and Ex. B are the same