14.3: The fundamental Theorem for Line Integrals

14.4: Green's Theorem

- Conservative vector field. (Sect. 14.1)

DEFINITION 1. *A vector field* **F** *is called a* **conservative vector field** *if it is the gradient of some scalar function* f *s.t* $\mathbf{F} = \nabla f$. *In this situation* f *is called a* **potential function** *for* **F**.

$$\vec{F}(x,y,z) = \langle P(x,y,z),\ Q(x,y,z),\ R(x,y,z) \rangle$$

$$\nabla f(x,y,z) = \langle f_x,\ f_y,\ f_z \rangle$$

$$\left.\begin{aligned} f_x &= P(x,y,z) \\ f_y &= Q(x,y,z) \\ f_z &= R(x,y,z) \end{aligned}\right\} \Rightarrow f = ?$$

REMARK 2. Not all vector fields are conservative, but such fields do arise frequently in Physics.

`Illustration:` **Gravitational Field:** By Newton's Law of Gravitation the magnitude of the gravitational force between two objects with masses m and M is The gravitational force acting on the object at (x, y, z) is

$$|\mathbf{F}| = G\frac{mM}{r^2},$$

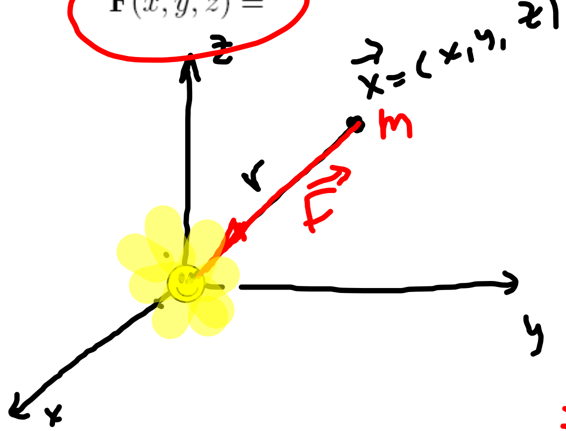
where $r = \sqrt{x^2 + y^2 + z^2}$ is the distance between the objects and G is the gravitational constant.

Let the position vector of the object with mass m be $\mathbf{x} = \langle x, y, z\rangle$. Then

$$r = \sqrt{x^2+y^2+z^2}, \qquad r^2 = x^2+y^2+z^2$$

Then the gravitational force acting on the object at $\mathbf{x} = \langle x, y, z\rangle$ is

$$\mathbf{F}(x, y, z) =$$



$$\vec{F}(x,y,z) = |\vec{F}| \cdot (-\hat{x}) =$$

$$= -|\vec{F}| \cdot \frac{\langle x, y, z\rangle}{(x^2+y^2+z^2)^{1/2}} =$$

$$= -\frac{GMm}{(x^2+y^2+z^2)^{3/2}}\langle x, y, z\rangle$$

$$\vec{F}(x,y,z) = \left\langle -\frac{GMm\,x}{(x^2+y^2+z^2)^{3/2}},\ -\frac{GMm\,y}{(x^2+y^2+z^2)^{3/2}},\ -\frac{GMm\,z}{(x^2+y^2+z^2)^{3/2}} \right\rangle$$

EXAMPLE 3. *Let*

$$f(x,y,z) = \frac{GmM}{\sqrt{x^2+y^2+z^2}}.$$

Find its gradient and answer the questions:

(a) *Is the gravitational field conservative?*

(b) *What is a potential function of the gravitational field?*

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

$$f_x = GmM \frac{\partial}{\partial x}\left((x^2+y^2+z^2)^{-\frac{1}{2}} \right) =$$

$$= GmM \cdot \left(-\frac{1}{2}\right)(x^2+y^2+z^2)^{-3/2} \cdot 2x =$$

$$= -\frac{GmM}{(x^2+y^2+z^2)^{3/2}} x$$

$$\nabla f = -\frac{GmM}{(x^2+y^2+z^2)^{3/2}} \langle x, y, z \rangle = \vec{F}(x,y,z)$$

Gravitational field

(a) The gravitational field is conservative.
(b) Its potential function is the given one up to a constant:

$$f(x,y,z) = \frac{GmM}{\sqrt{x^2+y^2+z^2}} + C$$

• **The fundamental Theorem for Line Integrals.** Recall Part 2 of the Fundamental Theorem of Calculus:

$$\int_a^b F'(x)\,\mathrm{d}x = F(b) - F(a),$$

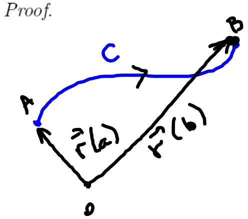
where F' is continuous on $[a,b]$.

Let C be a smooth curve given by $\mathbf{r}(t)$, $a \le t \le b$. Let f be a differentiable function of two or three variables and ∇f is continuous on C. Then

dot product

$$\int_C \nabla f \cdot \mathrm{dr} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)). = f(B) - f(A)$$

Proof.



$C: \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$a \le t \le b$

$\nabla f = \langle f_x, f_y, f_z \rangle$

$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt =$$

dot product

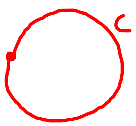
$$= \int_a^b \langle f_x(\vec{r}(t)), f_y(\vec{r}(t)), f_z(\vec{r}(t)) \rangle \cdot \langle x'(t), y'(t), z'(t) \rangle\,dt$$

$$= \int_a^b \left(f_x(\vec{r}(t))\,x'(t) + f_y(\vec{r}(t))\,y'(t) + f_z(\vec{r}\,'(t))\,z'(t) \right) dt =$$

Chain Rule $\displaystyle\int_a^b \frac{d f(\vec{r}(t))}{dt}\,dt =$ Fund. Theorem of Calculus

$$= f(\vec{r}(b)) - f(\vec{r}(a))$$

REMARK 4. If C is a closed curve then $\Rightarrow \vec{r}(a) = \vec{r}(b)$

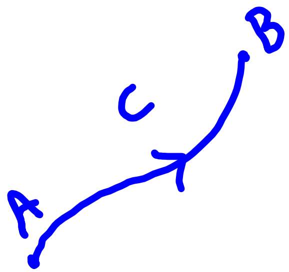


$$\oint_C \nabla f\, d\vec{r} = 0$$

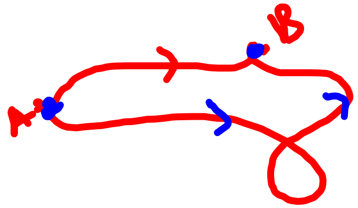
COROLLARY 5. *If F is a conservative vector field and C is a curve with initial point A and terminal point B then:*

$\Downarrow$

$$\boxed{\text{there exist a scalar function } f \text{ (potential) s.t. } \vec{F} = \nabla f}$$



$$\text{work} = \int_{\overset{\frown}{AB}} \vec{F}\, d\vec{r} = \int_{\overset{\frown}{AB}} \nabla f\, d\vec{r} \overset{\text{fund. theorem}}{=} f(B) - f(A)$$



EXAMPLE 6. *Find the work done by the gravitational field*

$$\mathbf{F}(x,y,z) = -\frac{GmM}{(x^2+y^2+z^2)^{3/2}}\langle x,y,z\rangle$$

in moving a particle with mass m from the point $(1,2,2)$ to the point $(3,4,12)$ along a piecewise-smooth curve C.

By Ex. 3 $\vec{F}$ is conservative and its potential is

$$f(x,y,z) = \frac{GmM}{\sqrt{x^2+y^2+z^2}}$$

$\Rightarrow \vec{F} = \nabla f$

$$W = \int_C \vec{F}\,d\vec{r} = \int_C \nabla f \cdot d\vec{r} = \int_{(1,2,2)}^{(3,4,12)} \nabla f \cdot d\vec{r} \overset{\text{FT + Corollary}}{=}$$

$$= f(3,4,12) - f(1,2,2) =$$

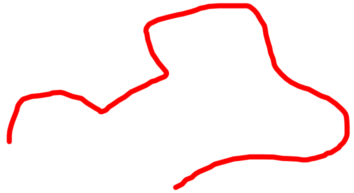
$$= \frac{GmM}{\sqrt{3^2+4^2+12^2}} - \frac{GmM}{\sqrt{1^2+2^2+3^2}} = GmM\left(\frac{1}{13} - \frac{1}{3}\right) =$$

$$= -\frac{10}{39}GmM$$

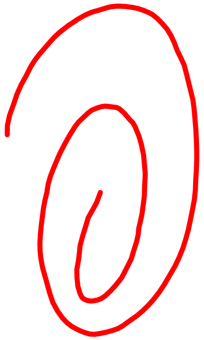
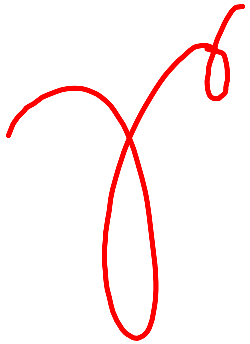
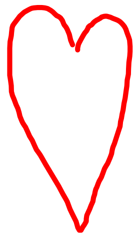
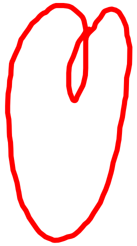
Note: $\int_A^B \vec{F}\,d\vec{r} = -\int_B^A \vec{F}\,d\vec{r}$

Notations And Definitions:

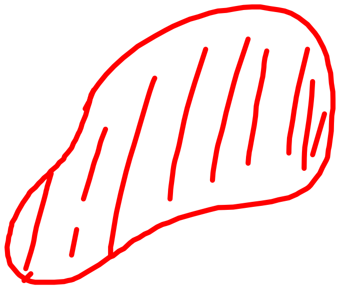
DEFINITION 7. *A piecewise-smooth curve is called a* **path**.



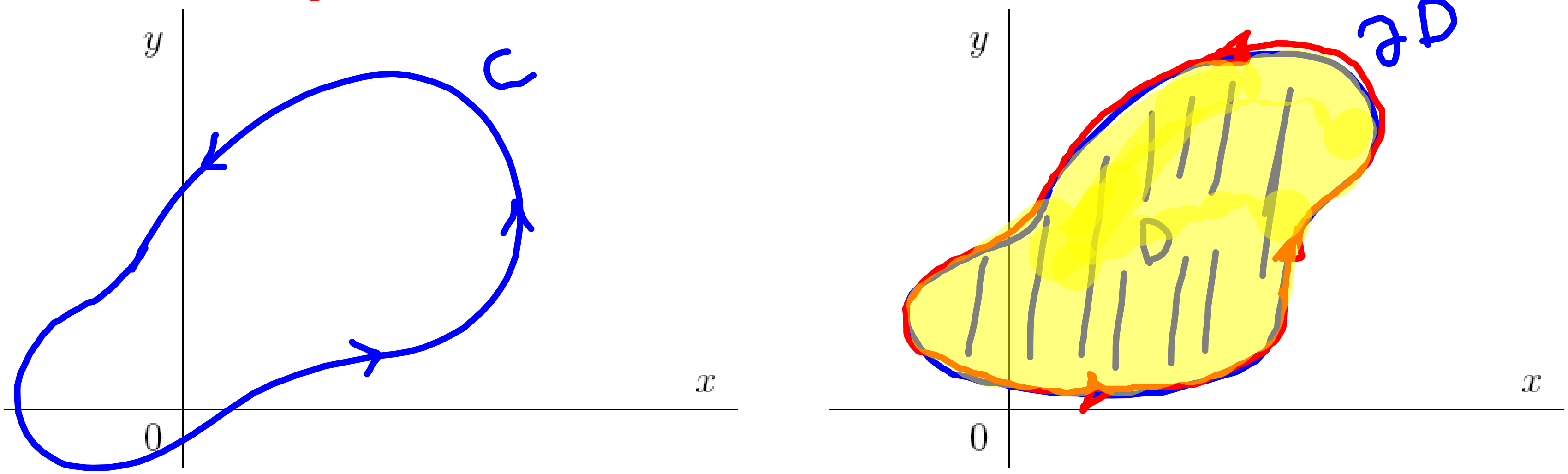
- **Types of curves:**

simple not closed	not simple not closed	simple closed	not simple, closed
			

- **Types of regions:**

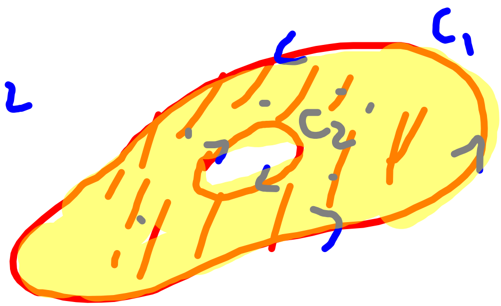
simply connected	not simply connected
	

- **Convention:** The **positive orientation** of a simple closed curve C refers to a single *counterclockwise* traversal of C. If C is given by $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j}, a \le t \le b$, then the region D bounded by C is always on the left as the point $\mathbf{r}(t)$ traverses C.



- The positively oriented boundary curve of D is denoted by ∂D.

$\partial D = C_1$ & C_2

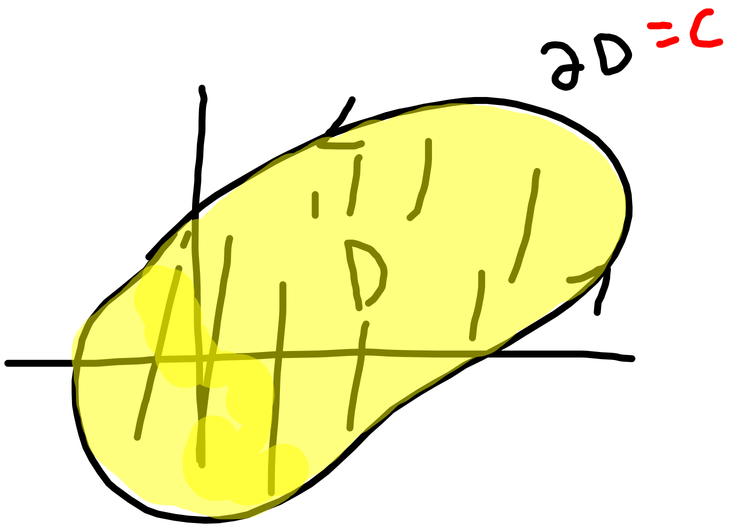


•**GREEN's THEOREM**: *Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If $P(x,y)$ and $Q(x,y)$ have continuous partial derivatives on an open region that contains D, then*

$$\oint_{\partial D} \vec{F} \, d\vec{r} = \oint_{\partial D} P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A.$$

where $\vec{F} = \langle P, Q \rangle$

Important that C is closed



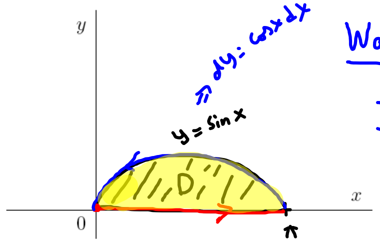
Note, if C is <u>not</u> positively oriented then

$$\oint_C \vec{F} \, d\vec{r} = -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

EXAMPLE 8. *Evaluate:*

$$I = \oint_{C = \partial D} \overbrace{e^x(1-\cos y)}^{P}\,dx \overbrace{- e^x(1-\sin y)}^{Q}\,dy$$

where C is the boundary of the domain $D = \{(x,y) : 0 \le x \le \pi, 0 \le y \le \sin x\}$.



Way 1

$$I = \int_0^\pi \left(e^x(1-\cos\pi)\right) dx + 0 +$$

$$+ \int_0^\pi e^x\left(1 - \cos(\sin x)\right) - e^x\left(1 - \sin(\sin x)\right)\cos x \, dx$$

We don't know how to find an antiderivative.

2nd way

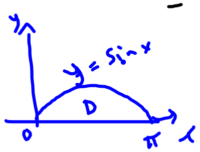
Since C is closed we can apply Green's Theorem:

$$P = e^x(1-\cos y) \Rightarrow \frac{\partial P}{\partial y} = e^x \sin y$$

$$Q = -e^x(1-\sin y) \Rightarrow \frac{\partial Q}{\partial x} = -e^x(1-\sin y) = -e^x + e^x \sin y$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -e^x + \cancel{e^x \sin y} - \cancel{e^x \sin y} = -e^x$$

By GT: $$I = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA =$$



$$= \iint_D -e^x \, dA = -\int_0^\pi \int_0^{\sin x} e^x \, dy\, dx =$$

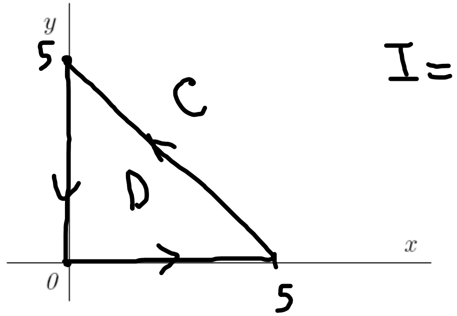
$$= -\int_0^\pi e^x \sin x \, dx =$$

$$= -\frac{1}{2} e^x (\sin x - \cos x) \Big|_0^\pi = -\frac{1}{2}(e^\pi + 1)$$

EXAMPLE 9. *Let C be a triangular curve consisting of the line segments from $(0,0)$ to $(5,0)$, from $(5,0)$ to $(0,5)$, and from $(0,5)$ to $(0,0)$. Evaluate the following integrals:*

(a) $I_1 = \oint_C (x^2y + \frac{1}{2}y^2)\,dx + (xy + \frac{1}{3}x^3 + 3x)\,dy$

C is closed ⇒ use Green's Theorem



$$I = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

where $\partial D = C$

$P = x^2 y + \frac{1}{2}y^2$

$Q = xy + \frac{1}{3}x^3 + 3x$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \cancel{y} + \cancel{x^2} + 3 - (\cancel{x^2} + \cancel{y}) = 3$$

$$I = \iint_D 3\,dA = 3\iint_D dA = 3 \text{ Area } D = 3\,\frac{5^2}{2} = \boxed{\frac{75}{2}}$$

(b) $I_2 = \oint_C (\overbrace{x^2y + \frac{1}{2}y^2 + e^{x\sin x}}^{P})\,dx + (\overbrace{xy + \frac{1}{3}x^3 + x - 4\arctan(e^y)}^{Q})\,dy$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = xy + \frac{1}{3}x^3 + 1 - (x^2 + y) = 1$$

$$I_2 = \iint_D 1\,dA = A(D) = \frac{25}{2}$$

(c) $I_3 = \oint_C (x^2y + \frac{1}{2}y^2 - 55\arcsin(\sec x))\,dx + (12y^5\cos y^3 + xy + \frac{1}{3}x^3 + x)\,dy$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 \Rightarrow I_3 = \frac{25}{2}$$

I_3 = Area of D

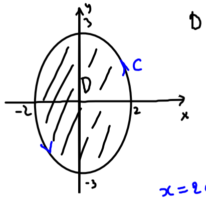
•**Application: Computing areas.**

$\boxed{\begin{matrix} P=0 \\ Q=x \end{matrix}}$ $\quad$ $\boxed{\begin{matrix} P=-y \\ Q=0 \end{matrix}}$ $\quad$ $\boxed{\begin{matrix} P=-\frac{y}{2} \\ Q=\frac{x}{2} \end{matrix}}$

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=1$$

$$A(D)=\boxed{\oint_{\partial D} x\,\mathrm{d}y}=\boxed{-\oint_{\partial D} y\,\mathrm{d}x}=\boxed{\frac{1}{2}\oint_{\partial D} x\,\mathrm{d}y-y\,\mathrm{d}x.}$$

EXAMPLE 10. *Find the area enclosed by the ellipse* $\frac{x^2}{4}+\frac{y^2}{9}=1.$ $= C = \partial D$



$D=\{(x,y): \frac{x^2}{4}+\frac{y^2}{9}\le 1\}$

$A(D)=\frac{1}{2}\oint_{\partial D} x\,dy-y\,dx$

Parametrize ∂D: $\underbrace{\left(\frac{x}{2}\right)}_{=\cos t}^2+\underbrace{\left(\frac{y}{3}\right)}_{=\sin t}^2=1$

$x=2\cos t,\quad y=3\sin t,\quad 0\le t\le 2\pi$

$dx=-2\sin t\,dt\quad,\ dy=3\cos t\,dt$

$$A(D)=\frac{1}{2}\int_0^{2\pi}\left(2\cos t\ 3\cos t-3\sin t\,(-2\sin t)\right)dt=$$

$$=\frac{1}{\cancel{2}}\cdot\cancel{2}\cdot 3\int_0^{2\pi}(\underbrace{\cos^2 t+\sin^2 t}_{=1})\,dt=3\cdot 2\pi=\boxed{6\pi}$$

Note: The area bdd by ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is πab

Must know

SUMMARY: Let $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ *be a vector field on an open simply connected domain* D*. Suppose that* P *and* Q *have continuous partial derivatives through* D*. Then the facts below are equivalent.*

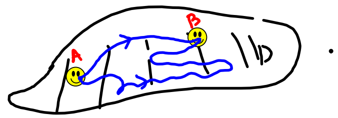
The field $\mathbf{F}$ is conservative on D	$\Longleftrightarrow$	There exists f s.t. $\nabla f = \mathbf{F}$

definition

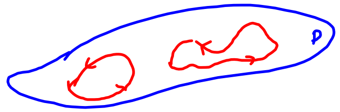
a potential

The field $\mathbf{F}$ is conservative on D	$\Longleftrightarrow$	$\int_{\overrightarrow{AB}} \mathbf{F}\cdot d\mathbf{r}$ is independent of path in D

$= f(B) - f(A)$



The field $\mathbf{F}$ is conservative on D	$\Longleftrightarrow$	$\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every closed curve C in D



$\vec{F} = P\hat{i} + Q\hat{j}$

The field $\mathbf{F}$ is conservative on D	$\Longleftrightarrow$	$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ throughout D

$\Rightarrow$ $\vec{F}$ is cons. $\Rightarrow$ there exists a potential f s.t.

$\nabla f = \vec{F} \Rightarrow \langle f_x, f_y\rangle = \langle P, Q\rangle \Rightarrow$

$$\Rightarrow \left.\begin{array}{l} f_x = P \\ f_y = Q \end{array}\right\} \Rightarrow \begin{array}{l} f_{xy} = (f_x)_y = P_y \\ \quad \shortparallel \qquad\qquad\quad \shortparallel \\ f_{yx} = (f_y)_x = Q_x \end{array}$$

$\Leftarrow$ If $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \Rightarrow$ for any closed C

$$\Rightarrow \oint_C \vec{F}\, d\vec{r} \overset{\text{Green Theorem}}{=} \iint_D \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{=0} dA = 0$$

$\Rightarrow \vec{F}$ is conservative

EXAMPLE 11. *Determine whether or not the vector field is conservative:*

(a) $\mathbf{F}(x,y) = \langle \overbrace{x^2+y^2}^{P}, \overbrace{2xy}^{Q} \rangle$.

$$\left.\begin{aligned} P_y &= 2y \\ Q_x &= 2y \end{aligned}\right\} \Rightarrow P_y = Q_x \Rightarrow \text{YES}$$

(b) $\mathbf{F}(x,y) = \langle \underbrace{x^2+3y^2+2}_{P}, \underbrace{3x+ye^y}_{Q} \rangle$

$$\left.\begin{aligned} P_y &= 6y \\ Q_x &= 3 \end{aligned}\right\} \Rightarrow P_y \neq Q_x \Rightarrow \text{NO}$$

EXAMPLE 12. *Given* $\mathbf{F}(x,y) = \overbrace{\sin y}^{P}\mathbf{i} + \overbrace{(x\cos y + \sin y)}^{Q}\mathbf{j}$.

(a) *Show that* $\mathbf{F}$ *is conservative.*

$P_y = \cos y$

$\Rightarrow P_y = Q_x \Rightarrow$ cons.

$Q_x = \cos y$

(b) *Find a function* f *s.t.* $\nabla f = \mathbf{F}$ (Find a potential function)

$f_x = P$

$f_y = Q$

$f_x = \sin y \Rightarrow f(x,y) = \int \sin y \, dx = x\sin y + C(y)$

$f_y = x\cos y + \sin y \Leftarrow \boxed{f(x,y) = x\sin y + C(y)}$

$(x\sin y + C(y))_y = x\cos y + \sin y$

$x\cancel{\cos}y + C'(y) = x\cancel{\cos}y + \sin y$

$C'(y) = \sin y$

$C(y) = \int \sin y \, dy = -\cos y + C$

$\boxed{f(x,y) = x\sin y - \cos y + C}$

(c) *Find the work done by the force field* $\mathbf{F}$ *in moving a particle from the point* $(3,0)$ *to the point* $(0,\pi/2)$

$$W = \int_{(3,0)}^{(0,\pi/2)} \vec{F}\cdot d\vec{r} \underset{\nabla f = \vec{F}}{\overset{f \text{ is cons.}}{=\!=}} \int_{(3,0)}^{(0,\frac{\pi}{2})} \nabla f \cdot d\vec{r} \overset{\text{Fundam. Theorem}}{=\!=}$$

$$= f\left(0,\tfrac{\pi}{2}\right) - f(3,0) =$$

$$= 0\sin\tfrac{\pi}{2} - \cos\tfrac{\pi}{2} + \cancel{C} - (3\sin 0 - \cos 0 + \cancel{C}) = 1$$

(d) *Evaluate* $\oint_C \mathbf{F}\,\mathrm{dr}$ *where* C *is an arbitrary path in* $\mathbb{R}^2$.

$\oint_C \vec{F} d\vec{r} = 0$ (since $\vec{F}$ is conservative)

EXAMPLE 13. *Given*

$$\vec{F}(x,y,z) = \mathbf{F} = \langle \overbrace{2xy^3 + z^2}^{P}, \overbrace{3x^2y^2 + 2yz}^{Q}, \overbrace{y^2 + 2xz}^{R} \rangle = \nabla f = \langle f_x, f_y, f_z \rangle$$

Find a function f s.t. $\nabla f = \mathbf{F}$

$f_x = 2xy^3 + z^2 \Rightarrow f(x,y,z) = \int (2xy^3 + z^2)\,dx$

$f_y = 3x^2y^2 + 2yz \Leftarrow \boxed{f(x,y,z) = x^2y^3 + xz^2 + C(y,z)}$

$f_z = y^2 + 2xz$

$$\Rightarrow \frac{\partial}{\partial y}\left(x^2y^3 + xz^2 + C(y,z)\right) = 3x^2y^2 + 2yz$$

$$\cancel{3x^2y^2} + 0 + \underbrace{\frac{\partial C(y,z)}{\partial y}} = \cancel{3x^2y^2} + \underbrace{2yz}$$

$$C(y,z) = \int 2yz\,dy + K(z) = y^2z + K(z)$$

Update f:

$$\boxed{f(x,y,z) = x^2y^3 + xz^2 + y^2z + K(z)}$$

$$0 + 2\cancel{x}z + \cancel{y^2} + K'(z) = \cancel{y^2} + 2\cancel{xz}$$

$$K'(z) = 0$$

$$K(z) = \text{constant}$$

Update f:

$$\underbrace{f(x,y,z) = x^2y^3 + xz^2 + y^2z + C}$$ potential function for $\vec{F}$

Note: The result means that $\vec{F}$ is conservative.