14.5: Curl and Divergence

Introduce the vector differential operator ∇ as

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

If $\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ is a vector field on R^3 and the partial derivatives of P, Q, R all exist, then the **curl** of $\mathbf{F}$ is the *vector field* on R^3 defined by

$$\underbrace{\mathrm{curl}\mathbf{F}}_{\text{vector field}} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \hat{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix} - \hat{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix} + \hat{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - \hat{j}\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + \hat{k}\underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{\text{relates to Green's theorem}}$$

EXAMPLE 1. *Find the curl of the vector field*

$$\mathbf{F}(x,y,z) = \langle xy, x^2, yz \rangle.$$

$$\mathrm{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & x^2 & yz \end{vmatrix} = \hat{i}\underbrace{\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & yz \end{vmatrix}}_{\frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(x^2)} - \cdots$$

$$= \hat{i}(z-0) - \hat{j}(0-0) + \hat{k}(2x-x)$$

$$= \langle z, 0, x \rangle$$

Question *What is the curl of a two-dimensional vector field*

$$\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j} \text{ ?}$$

Answer:

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \langle 0, 0, \underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{\text{relates to Green's theorem}} \rangle = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

CONCLUSION: Green's Theorem in vector form:

$$\hat{k}\cdot\hat{k} = 1$$

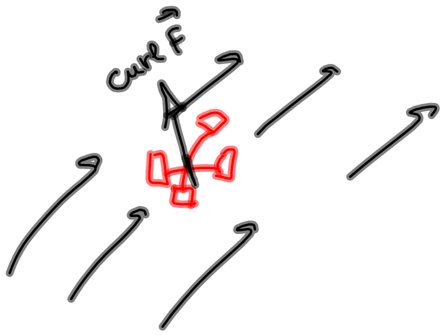
$$\oint_{\partial D} \mathbf{F}\cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

double integral

$$= \iint_D \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}}_{\operatorname{curl}\vec{F}} \cdot \underbrace{\hat{k}\, dA}_{d\vec{S}}$$

$$= \iint_D \operatorname{curl}\vec{F}\cdot d\vec{S}$$

surface integral

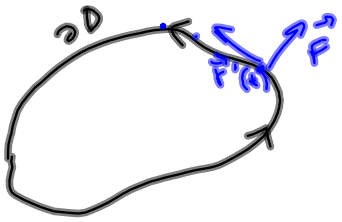


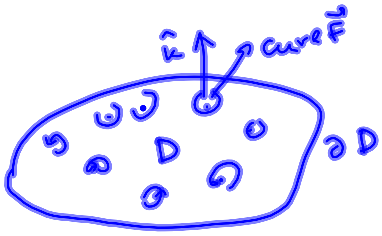
$\vec F = \rho \vec v$

$|\text{curl}\,\vec F| =$ angular velocity

Green's theorem:

$$\oint_{\partial D} \vec F \cdot \underbrace{d\vec r}_{\vec r\,'(t)\,dt} = \iint_D \underbrace{\text{curl}\,\vec F \cdot \hat k}_{\text{microcirculation}}\, dA$$





Circulation of $\vec F$ around ∂D

THEOREM 2. *If a function $f(x, y, z)$ has continuous partial derivatives of second order then*

$$\operatorname{curl}(\nabla f) = 0.$$

Proof. $\operatorname{curl}(\underbrace{\nabla f}) = \nabla \times (\nabla f) = (\underbrace{\nabla \times \nabla}_{= 0}) f = \vec{0}$

COROLLARY 3. $\underbrace{\textit{If } \mathbf{F} \textit{ is conservative,}}$ *then* $\operatorname{curl}\mathbf{F} = 0$.

$\Downarrow$

there exists potential f such that

$\vec{F} = \nabla f \Rightarrow \operatorname{curl} \vec{F} = \operatorname{curl}(\nabla f) = \vec{0}$

The proof of the Theorem below requires Stokes' Theorem (Section 14.8).

THEOREM 4. *If* $\mathbf{F}$ *is a vector field defined on* R^3 *whose component functions have continuous partial derivatives and* $\mathrm{curl}\mathbf{F} = 0$, *then* $\mathbf{F}$ *is a conservative vector field.*

EXAMPLE 5. *Let* $\mathbf{F}(x, y, z) = \langle x^9, y^9, z^9 \rangle$.

(a) *Show that* $\mathbf{F}$ *is conservative.*

$$\operatorname{curl} \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^9 & y^9 & z^9 \end{vmatrix} = \langle 0, 0, 0 \rangle = \vec{0} \Rightarrow \vec{F} \text{ is conserv.}$$

(b) *Find a function* f *s.t.* $\nabla f = \mathbf{F}$.

$f_x = x^9$

$f_y = y^9$

$f_z = z^9$

$$f(x, y, z) = \frac{x^{10}}{10} + \frac{y^{10}}{10} + \frac{z^{10}}{10} + C$$

(Check!)

(c) *Evaluate* $\int_{(1,0,1)}^{(-1,-1,-1)} \mathbf{F} \cdot d\mathbf{r}$. $= \int_{(1,0,1)}^{(-1,-1,-1)} \nabla f \cdot dr \overset{\text{FTLI}}{=} f(-1,-1,-1) - f(1,0,1)$

$$= \frac{3}{10} - \frac{2}{10} = \boxed{\frac{1}{10}}$$

If $\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ is a vector field on R^3 and the partial derivatives P_x, Q_y, R_z exist, then the **divergence of F** is the *scalar field* on defined by

$$\mathrm{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle .$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$ scalar function

EXAMPLE 6. *Find the divergence of the vector field*

$$\mathbf{F}(x,y,z) = \langle \overset{P}{\sin(xyz)}, \overset{Q}{x^2}, \overset{R}{yz} \rangle .$$

$$\mathrm{div}\,\vec{F} = \frac{\partial}{\partial x}(\sin(xyz)) + \frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial z}(yz)$$

$$= yz\cos(xyz) + 0 + y = yz\cos(xyz) + y$$

THEOREM 7. *If the components of a vector field* $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ *has continuous partial derivatives of second order then*

$$\text{div curl } \mathbf{F} = 0.$$

Proof.

$$\operatorname{div}(\operatorname{curl} \vec{F}) = \underbrace{\nabla \cdot (\nabla \times \vec{F})}_{\text{scalar triple product of coplanar vectors}} = 0$$

EXAMPLE 8. *Is there a vector field* $\mathbf{G}$ *on* $\mathbb{R}^3$ *s.t.* curl $G = \langle yz, xyz, zy \rangle$*?*

If such $\vec{G}$ exists then by Theorem 7

we would have

$$\operatorname{div}(\operatorname{curl} \vec{G}) = 0$$

On the other hand, $\operatorname{div}(\operatorname{curl} \vec{G}) = \operatorname{div}\langle yz, xyz, zy \rangle$

$$= \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(zy)$$

$$= 0 + xz + y \neq 0$$

for all (x, y, z)

NO