14.7: Surface Integrals

Problem: Find the **mass** of a thin sheet (say, of aluminum foil) which has a shape of a surface S and the density (mass per unit area) at the point (x, y, z) is $\rho(x, y, z)$.

Solution:

If $\rho = \text{const} \Rightarrow m = \rho\,(\text{surface area}) = \rho \iint_S dS$

If $\rho \neq \text{const} \Rightarrow m = \iint_S \rho(x, y, z)\, dS$

If S is given by $\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$, $(u,v) \in D$, then the **surface integral of f over the surface S** is:

$$\iint_S f(x,y,z)\,\mathrm{d}S = \iint_D f(\mathbf{r}(u,v))|\mathbf{N}(u,v)|\,\mathrm{d}A =$$

$$= \iint_D f\left(x(u,v), y(u,v), z(u,v)\right) |\vec{r}_u \times \vec{r}_v| \, du\,dv$$

double integral

$$dS = |\vec{r}_u \times \vec{r}_v| \, du\,dv$$

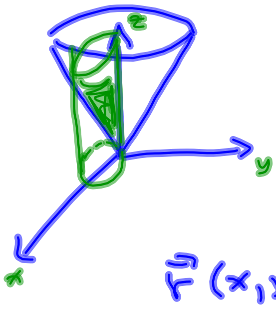
or $$dS = |\vec{N}| \, dA$$

EXAMPLE 1. *Find the mass of a thin funnel in the shape of a cone* $z = \sqrt{x^2+y^2}$ *inside the cylinder* $x^2+y^2 \le 2x$, *if its density is a function* $\rho(x,y,z) = x^2+y^2+z^2$.

S graph

$(x-1)^2+y^2 \le 1$

$$m = \iint_S \rho\, dS = \iint_S (x^2+y^2+z^2)\, dS$$



Parameterize S:

$x = x$, $y = y$, $z = \sqrt{x^2+y^2}$

$$\underbrace{\vec{r}(x,y) = \langle x, y, \sqrt{x^2+y^2} \rangle}$$

Find $\vec{r}_x \times \vec{r}_y$ and so on . . .

. . . . .

$(x,y) \in D = \{x^2+y^2 \le 2x\}$

shorter way

normal vector to the graph $z = f(x,y)$ is $\vec{r}_x \times \vec{r}_y = \vec{N}(x,y) = \langle f_x, f_y, -1 \rangle$

$|\vec{N}(x,y)| = \sqrt{f_x^2 + f_y^2 + 1}$

$$\vec{N}(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1 \right\rangle$$

$$|\vec{N}(x,y)| = \sqrt{\underbrace{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}}_{1} + 1} = \sqrt{2}$$

$$m = \iint_S (x^2+y^2+z^2)\, dS = \iint_D (x^2+y^2+\overbrace{x^2+y^2}^{z^2})\overbrace{\sqrt{2}}^{\vec{r}_x \times \vec{r}_y}\, dx\,dy$$

$D = \{(x,y) : x^2+y^2 \le 2x\}$

use polar coord.

$$= \iint_{D^*} 2\sqrt{2}\, r^2\, r\, dr\, d\theta$$

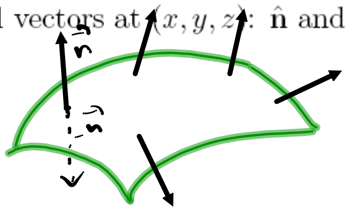
$D^* = \{(r,\theta) : 0 \le r \le 2\cos\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\}$

$$2\sqrt{2} \int_{-\frac{\pi}{2}}^{\pi/2} \int_0^{2\cos\theta} r^3\, dr\, d\theta = 2\sqrt{2} \int_{-\frac{\pi}{2}}^{\pi/2} \frac{r^4}{4} \Big|_0^{2\cos\theta} d\theta$$

$$= \frac{\sqrt{2}}{2} \int_{-\pi/2}^{\pi/2} 2^4 \cos^4\theta\, d\theta = \ldots = 3\sqrt{2}\,\pi$$

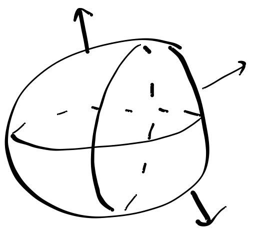
• **Oriented surfaces.** We consider only two-sided surfaces.

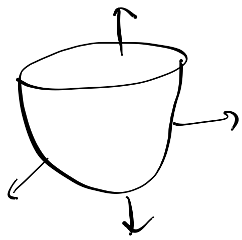
Let a surface S has a tangent plane at every point (except at any boundary points). There are two unit normal vectors at (x, y, z): $\hat{\mathbf{n}}$ and $-\hat{\mathbf{n}}$.



If it is possible to choose a unit normal vector $\hat{\mathbf{n}}$ at every point (x, y, z) of a surface S so that $\hat{\mathbf{n}}$ varies continuously over S, then S is called **oriented surface** and the given choice of $\hat{\mathbf{n}}$ provides S with an **orientation**. There are two possible orientations for any orientable surface:

`Convention:` *For closed surfaces the positive orientation is outward.*





• **Surface integrals of vector fields.**

DEFINITION 2. *If* **F** *is a continuous vector field defined on an oriented surface* S *with unit normal vector* $\hat{\mathbf{n}}$, *then the* **surface integral of F over** S *is*

$$\iint_S \mathbf{F} \cdot \mathrm{d}\vec{\mathbf{S}} = \iint_S \mathbf{F} \cdot \overbrace{\underbrace{\hat{\mathbf{n}}}_{\text{unit}}\,\mathrm{d}S.}^{d\vec{S}}$$

This integral is also called the **flux** *of* **F** *across* S.

Note that if S is given by $\mathbf{r}(u, v)$, $(u, v) \in D$, then

$$\hat{\mathbf{n}} = \frac{\mathbf{n}}{|\mathbf{n}|} = \frac{\vec{n}(u,v)}{|\vec{n}(u,v)|} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

and

$$d\vec{\mathbf{S}} = \hat{n}\,\underbrace{dS} = \hat{n}(u,v) \cdot |\vec{n}(u,v)|\,du\,dv$$

$$= \frac{\vec{n}(u,v)}{\cancel{|\vec{n}(u,v)|}} \cdot \cancel{|\vec{n}(u,v)|}\,du\,dv$$

$$= \vec{n}(u,v)\,du\,dv$$

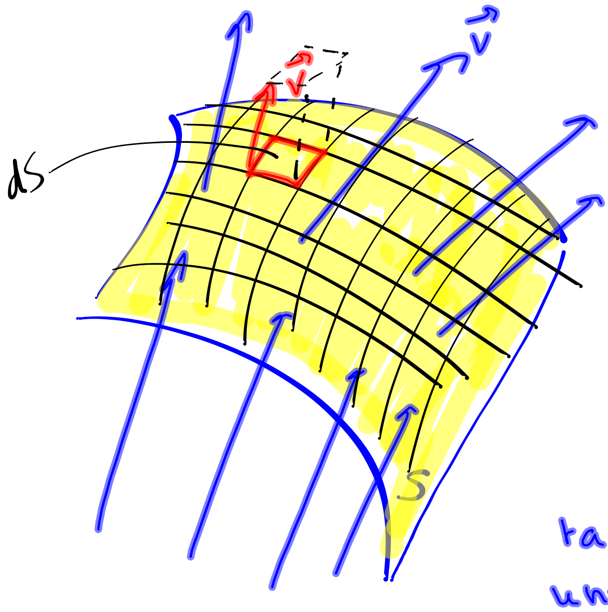
$$= \pm\,\vec{r}_u \times \vec{r}_v\,du\,dv$$

Finally,

$$\iint_S \mathbf{F} \cdot \mathrm{d}\vec{\mathbf{S}} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\pm)(\vec{r}_u \times \vec{r}_v)\,du\,dv$$

choose correct orientation of $\vec{n} = \pm\,\vec{r}_u \times \vec{r}_v$ depending on problem

$$\text{Flux} \qquad \iint_S \vec F\cdot d\vec S = \iint_S \vec F\cdot \hat n\, dS = \iint_D \vec F(\vec r(u,v))\cdot \vec n(u,v)\, du\, dv$$



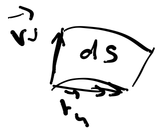
Fluid with density $\rho(x,y,z)$ and velocity field $\vec V(x,y,z)$ through S

$$\vec F = \rho \vec V$$

$$\left[\frac{kg}{m^3}\cdot\frac{m}{s}\right] = \left[\frac{kg}{m^2\cdot s}\right]$$

$\Downarrow$

rate of flow = mass per unit area and unit time

Volume of the fluid crosses dS per unit time is approximately equal to volume of the 

parallelepiped spanned by $\vec v$, $\vec r_u$, $\vec r_v$

$$\text{Volume} \approx \vec v\cdot(\vec r_u \times \vec r_v)$$

$$\text{mass} = \rho\cdot\text{Volume} = \underbrace{\rho\,\vec v}_{\vec F}\cdot(\vec r_u\times\vec r_v) = \vec F\cdot\underbrace{(\vec r_u\times\vec r_v)}_{\vec n(u,v)}$$

= Flux through dS

$$= \vec F\cdot\vec n$$

$$\text{Total Flux} = \iint_S \vec F\cdot\underbrace{\hat n\, dS}_{d\vec S} = \iint_S \vec F\cdot d\vec S$$

EXAMPLE 3. *Find the flux of the vector field*

$$\mathbf{F} = \langle x^2, y^2, z^2 \rangle$$

across the surface

$$S = \{z^2 = x^2 + y^2, 0 \le z \le 2\}.$$

$$\text{Flux} = \iint_S \vec{F} \cdot d\vec{S}$$

① Parameterize S

way 1 $\quad z = \sqrt{x^2+y^2}$

$\vec{r}(x,y) = \{x, y, \sqrt{x^2+y^2}\}$

$D = \underbrace{\{ x^2+y^2 \le 4 \}}_{\text{projection of } S \text{ onto the xy-plane}}$

. . . .

way 2 use polar coord.

$x = u\cos v, \quad y = u \sin v, \quad z = u$

$\vec{r}(u,v) = \langle u\cos v, u \sin v, u \rangle$

$D = \{(u,v) : 0 \le v \le 2\pi, 0 \le u \le 2\}$

② Find normal $\vec{n}(u,v)$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \langle -u\cos v, -u\sin v, u \rangle$$

$= u\cos^2 v + u\sin^2 v$

$\vec{n}(u,v) = \pm \vec{r}_u \times \vec{r}_v$

$= \pm \langle -u\cos v, -u \sin v, u \rangle$

From the picture $\vec{n}$ is pointed downward

$\Rightarrow$ the z-component of $\vec{n}$ is negative

$\vec{n}(u,v) = \langle u\cos v, u\sin v, -u \rangle$

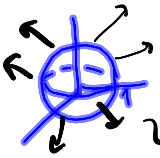
③ Find dot product

$\vec{F}(\vec{r}(u,v)) \cdot \vec{n}(u,v) =$

$= \langle$

EXAMPLE 4. *Evaluate* $I = \underbrace{\iint_S \mathbf{F}\cdot d\mathbf{S}}_{\text{Flux}}$ *where* $\mathbf{F} = \langle z, y, x\rangle$ *and* S *is* $\underbrace{\textit{the unit sphere}}_{\text{closed}}$ *centered at the origin.*

$\Downarrow$ $\vec{n}$ is outward



Parameterize $S: x^2+y^2+z^2=1$

Use spherical coordinates: $\rho^2 = 1 \Rightarrow \rho = 1$

$$\vec{r}(\theta,\varphi) = \langle \underbrace{\sin\varphi\cos\theta}_{x},\ \underbrace{\sin\varphi\sin\theta}_{y},\ \underbrace{\cos\varphi}_{z} \rangle$$

$D = \{(\varphi,\theta) : 0 \le \varphi \le \pi,\ 0 \le \theta \le 2\pi\}$

$$r_\theta \times r_\varphi = \begin{vmatrix} i & j & k \\ -\sin\varphi\sin\theta & \sin\varphi\cos\theta & 0 \\ \cos\varphi\cos\theta & \cos\varphi\sin\theta & -\sin\varphi \end{vmatrix}$$

$$= \langle -\sin^2\varphi\cos\theta,\ -\sin^2\varphi\sin\theta,\ -\sin\varphi\cos\varphi\underbrace{(\sin^2\theta+\cos^2\theta)}_{=1} \rangle$$

$$\vec{n}(\theta,\varphi) = \pm\langle -\sin^2\varphi\cos\theta,\ -\sin^2\varphi\sin\theta,\ -\sin\varphi\cos\varphi \rangle$$

$dS = \vec{n}\, dA$

$$\vec{F}(\vec{r}(\theta,\varphi))\cdot\vec{n}(\theta,\varphi)$$

$$= \langle \cos\varphi,\ \sin\varphi\sin\theta,\ \sin\varphi\cos\theta \rangle \cdot \langle \sin^2\varphi\cos\theta,\ \sin^2\varphi\sin\theta,\ \sin\varphi\cos\varphi \rangle$$

$$= \sin^2\varphi\cos\varphi\cos\theta + \sin^3\varphi\sin^2\theta + \sin^2\varphi\cos\varphi\cos\theta$$

$$= 2\sin^2\varphi\cos\varphi\cos\theta + \sin^3\varphi\sin^2\theta$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(\theta, \varphi)) \cdot \vec{n}(\theta, \varphi)\, dA$$

$$= \int_0^{\pi} \int_0^{2\pi} \left(2 \sin^2\varphi \cos\varphi \cos\theta + \sin^3\varphi \sin^2\theta\right) d\theta\, d\varphi$$

$$= 2\int_0^{\pi} \sin^2\varphi \cos\varphi \, d\varphi \underbrace{\int_0^{2\pi} \cos\theta \, d\theta}_{= 0} + \underbrace{\int_0^{\pi} \sin^3\varphi \, d\varphi}_{} \int_0^{2\pi} \sin^2\theta \, d\theta$$

$$= \int_0^{\pi} (1-\cos^2\varphi) \sin\varphi \, d\varphi \int_0^{\pi} \frac{1-\cos 2\theta}{2} \, d\theta$$

$$= \dots = \frac{4\pi}{3}$$