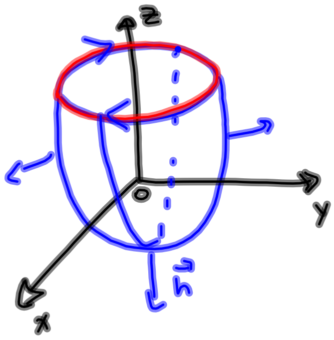
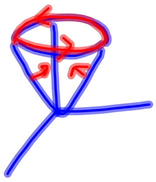
14.8: STOKES' THEOREM

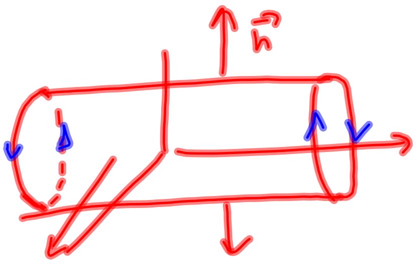
Stokes' Theorem can be regarded as a 3-dimensional version of Green's Theorem:

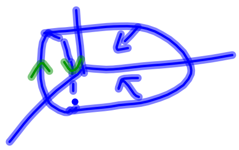
$$\oint_{C=\partial D} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A = \iint_D \operatorname{curl}\mathbf{F}\cdot \mathbf{k}\,\mathrm{d}A.$$

Let S be an oriented surface with unit normal vector $\hat{\mathbf{n}}$ and with the boundary curve C (which is a space curve).









The orientation on S induces the **positive orientation of the boundary curve** C: if you walk in the positive direction around C with your head pointing in the direction of $\hat{\mathbf{n}}$, then the surface will always be on your left.

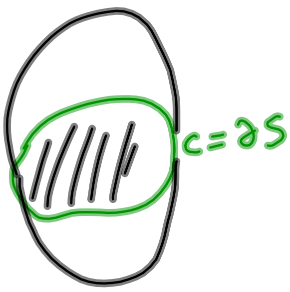
The positively oriented boundary curve of an oriented surface S is often written as ∂S.

Stokes' Theorem: *Let S be an oriented piece-wise-smooth surface that is bounded by a simple closed, piecewise smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whos components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then*

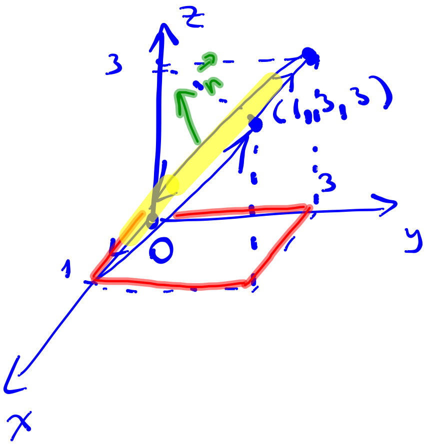
$$\oint_C \mathbf{F} \cdot \, \mathrm{d}\mathbf{r} = \iint_S \mathrm{curl}\mathbf{F} \cdot \, \mathrm{d}\mathbf{S},$$

or

$$\iint_S \mathrm{curl}\mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S = \oint_{\partial S} \mathbf{F} \cdot \, \mathrm{d}\mathbf{r}.$$



EXAMPLE 1. *Find the work performed by the forced field* $\mathbf{F}(x,y,z) = \langle 3x^8, 4xy^3, y^2x\rangle$ *on a particle that traverses the curve* C *in the plane* $z=y$ *consisting of 4 line segments from* $(0,0,0)$ *to* $(1,0,0)$*, from* $(1,0,0)$ *to* $(1,3,3)$*, from* $(1,3,3)$ *to* $(0,3,3)$*, and from* $(0,3,3)$ *to* $(0,0,0)$.



$$\text{Work} = \oint_C \vec{F}\cdot d\vec{r}$$

C is closed path ⇒ C is boundary of a surface → we can apply Stokes' Theorem

Consider C as edge (= boundary) of portion of the plane $z=y$

$S:\ z=y$ where $0\le x\le 1$, $0\le y\le 3$

$$\text{Work} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$$ surface integral (flux)

$$\text{curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^8 & 4xy^3 & y^2x \end{vmatrix} = \langle 2yx, -y^2, 4y^3\rangle$$

Parameterize S

S: $x = x$, $y = y$, $\underbrace{z = y}$

$D = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le 3\}$

$\vec{n}(x, y) = \pm \langle z_x, z_y, -1 \rangle$

$= \pm \langle 0, 1, -1 \rangle = \langle 0, -1, 1 \rangle$

$$\text{Work} = \iint_S \langle 2yx, -y^2, 4y^3 \rangle \cdot \underbrace{d\vec{S}}_{\vec{n}(x,y)\, \underbrace{dA}_{dx\,dy}}$$

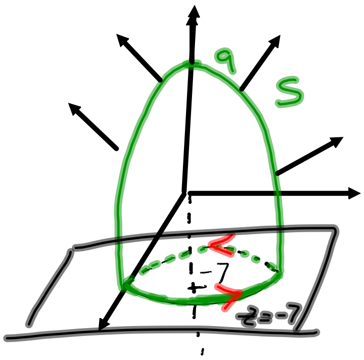
double integral

$$= \iint_D \langle 2yx, -y^2, 4y^3 \rangle \cdot \langle 0, -1, 1 \rangle \, dA$$

$$= \iint_D (0 + y^2 + 4y^3)\, dx\, dy = \int_0^1 \int_0^3 (y^2 + 4y^3)\, dy = \frac{y^3}{3} + y^4 \Big|_0^3$$

$$= \ldots = \boxed{252}$$

EXAMPLE 2. *Verify Stokes' Theorem* $\iint_S \text{curl}\vec{F}\cdot \mathrm{d}\vec{S} = \int_{\partial S} \vec{F}\cdot \mathrm{d}\vec{r}$ *for the vector field* $\vec{F} = \langle 3y, 4z, -6x\rangle$ *and the paraboloid* $z = 9 - x^2 - y^2$ *that lies above the plane* $z = -7$ *and oriented upward. Be sure to check and explain the orientations.*



Solution: Use the following steps:

•Parametrize the boundary circle ∂S and compute the line integral.

$$\begin{cases} z = 9 - x^2 - y^2 \\ z = -7 \end{cases} \Rightarrow \quad 9 - x^2 - y^2 = -7 \qquad \partial S: \; x^2 + y^2 = 16, \; z = -7$$

$$\vec{r}(\theta) = \langle 4\cos\theta, 4\sin\theta, -7\rangle, \quad 0 \le \theta \le 2\pi$$

$$d\vec{r} = \vec{r}\,'(\theta)\,d\theta = \langle -4\sin\theta, 4\cos\theta, 0\rangle\, d\theta$$

$$\vec{F}(\vec{r}(\theta))\cdot d\vec{r} = \langle 3\cdot 4\sin\theta, 4\cdot(-7), -6\cdot 4\cos\theta\rangle \cdot \langle -4\sin\theta, 4\cos\theta, 0\rangle\, d\theta$$

$$= -48\sin^2\theta - 16\cdot 7\cos\theta$$

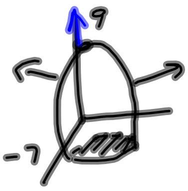
$$\int_{\partial S}\vec{F}\cdot d\vec{r} = \int_0^{2\pi} -48\sin^2\theta - 16\cdot 7\underbrace{\cos\theta}_{=0}\, d\theta = -48\int_0^{2\pi}\frac{1-\cos 2\theta}{2}\, d\theta$$

$$= -24\left[\underbrace{\int_0^{2\pi} 1\, d\theta}_{2\pi} - \underbrace{\int_0^{2\pi}\cos 2\theta\, d\theta}_{=0}\right] = -48\pi$$

•Paramtrize the surface of the paraboloid and compute the surface integral:

$$\iint_S \underbrace{\text{curl}\,\vec{F}}\cdot\underbrace{d\vec{S}} = \iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,dA$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & 4z & -6x \end{vmatrix} = \langle -4, 6, -3\rangle$$



$S:\quad z = 9 - x^2 - y^2,\quad -7 \le z \le 9$

Parameterization:

$x = x,\quad y = y,\quad z = 9 - x^2 - y^2,\quad D = \{x^2 + y^2 \le 16\}$

$$\vec{n}(x,y) = \pm\langle z_x, z_y, -1\rangle = -\langle -2x, -2y, -1\rangle = \langle 2x, 2y, 1\rangle$$

$$\iint_D \langle -4, 6, -3\rangle\cdot\langle 2x, 2y, 1\rangle\,dA$$

use polar coord.

$$= \underbrace{\iint_D (-8x + 12y - 3)\,dA}_{\text{double integral over disk}} = \iint_D -8x + 12y\,dA - 3\underbrace{\iint_D dA}_{\substack{\text{area of } D \\ 16\pi}} =$$

$$= 0 - 3\cdot 16\pi = \boxed{-48\pi}$$

$$\int_0^{2\pi}\int_0^4 (-8r\cos\theta + 12r\sin\theta)\,r\,dr\,d\theta$$

$$= \underbrace{\int_0^{2\pi}\cos\theta\,d\theta}_{=0}\int_0^4 -8r^2\,dr + \underbrace{\int_0^{2\pi}\sin\theta\,d\theta}_{=0}\int_0^4 12r^2\,dr$$

THEOREM 3. *If* $\mathbf{F}$ *is a vector field defined on* $\mathbb{R}^3$ *whose component functions have continuous partial derivatives and* $\text{curl}\mathbf{F} = \vec{\mathbf{0}}$, *then* $\mathbf{F}$ *is a conservative vector field.*

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = 0$$ for any closed path C in $\mathbb{R}^3$

If $\vec{F}$ is conservative then

SUMMARY: Let $\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ *be a continuous vector field in* $\mathbb{R}^3$.

There exists f s.t. $\nabla f = \mathbf{F}$ potential	$\Leftrightarrow$ $\mathbf{F}$ is conservative in $\mathbb{R}^3$ $\Leftrightarrow$	$\int_{\overset{\frown}{AB}} \mathbf{F}\cdot d\mathbf{r}$ is independent of path $= f(B) - f(A)$
$\text{curl}\mathbf{F} = \vec{\mathbf{0}}$	$\Leftrightarrow$ $\mathbf{F}$ is conservative in $\mathbb{R}^3$ $\Leftrightarrow$	$\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every closed curve C