14.9: The Divergence Theorem (GAUSS theorem)

Let E be a simple solid region with the boundary surface S (which is a closed surface.) Let S be positively oriented (i.e.the orientation on S is outward that is, the unit normal vector $\hat{\mathbf{n}}$ is directed outward from E).

Examples:

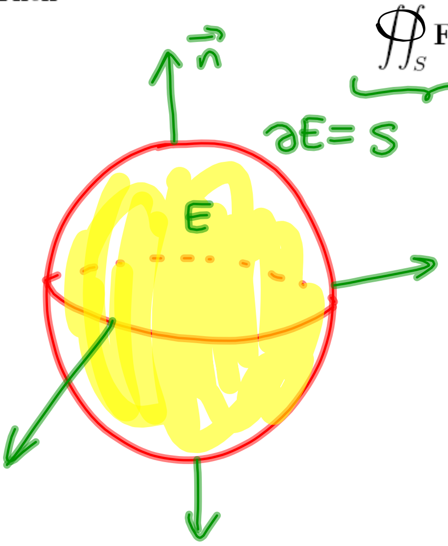
solid sphere $E: x^2+y^2+z^2 \leq 1$

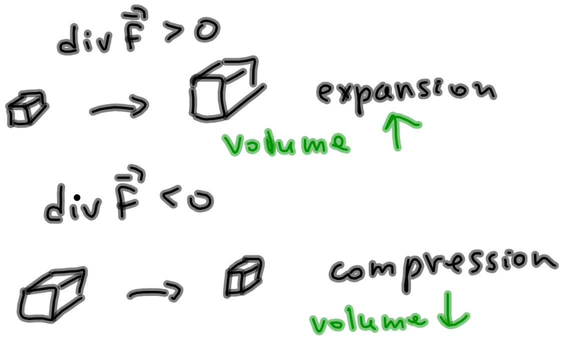
with boundary surface $S: x^2+y^2+z^2 = 1$

	boundary	# parameters
solid E (3 param.)	surface S (closed surface)	2 param.
surface S (2 param)	curve C	1 param
plane region D	curve C	1 param
closed surface S	no boundary	

The Divergence Theorem: Let E be a simple solid region whose boundary surface S has positive (outward) orientation. Let $\mathbf{F}$ be a continuous vector field on an open region that contains E. Then

$$\underbrace{\oiint_S \mathbf{F}\cdot d\mathbf{S}}_{= \text{Flux}} = \iiint_E \operatorname{div}\mathbf{F}\, dV.$$





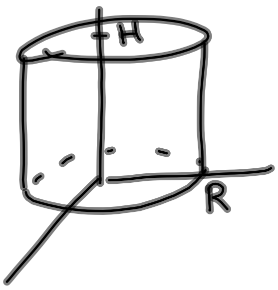
Divergence is the tendency of the vector field to diverge (to move toward) from the point.

$$\operatorname{div}\vec{F} = \text{const} \Rightarrow \underbrace{\oiint_S \vec{F}\cdot d\vec{s}}_{\text{mass}} = \operatorname{div}\vec{F} \underbrace{\iiint_V dV}_{\text{volume}}$$

$$\operatorname{div}\vec{F} = \frac{\oiint \vec{F}\cdot d\vec{s}}{\iiint_V dV} = \frac{\text{flux}}{\text{volume}}$$

Gauss Theorem says that divergence is the outgoing flux per volume solid cylinder

EXAMPLE 1. *Let* $E = \{(x, y, z) : x^2 + y^2 \leq R^2, 0 \leq z \leq H\}$. *Find the flux of the vector field* $\mathbf{F} = \langle 1 + x, 3 + y, z - 10\rangle$ *over* $\partial E = S$



$\partial E = S$ consists of portion of the cylinder $x^2 + y^2 = R^2$ between $z = 0$ & $z = H$ and two "Lids" on the planes $z = 0$ and $z = H$.

S is closed surface $\Rightarrow$ positive (outward) orientation

Way 1 Parameterize S (you need to parametrize 3 surfaces here) ....

$\operatorname{div} \vec{F} = 1 + 1 + 1 = 3$

Way 2 ☺ Flux $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV = 3 \underbrace{\iiint_E dV}_{\text{volume}}$

$= 3\pi R^2 H$

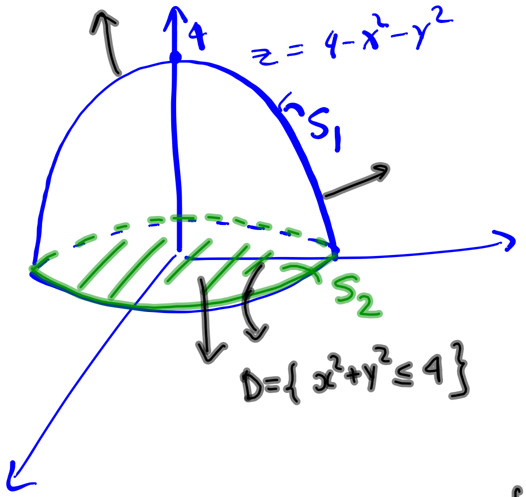
REMARK 2. If $\mathbf{F} = \left\langle \frac{x}{3}, \frac{y}{3}, \frac{z}{3} \right\rangle$ then $\operatorname{div} \vec{F} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$

$$\iint_{\partial E} \vec{F} \cdot d\vec{S} = \boxed{\iint_{\partial E = S} \left\langle \frac{x}{3}, \frac{y}{3}, \frac{z}{3} \right\rangle \cdot d\vec{S} = \iiint_E dV = \text{Volume of } E}$$

EXAMPLE 3. *Let E be the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane. Evaluate* $I = \iint_S \underbrace{\langle x^3, 2xz^2, 3y^2 z\rangle}_{\vec{F}} \cdot \mathrm{d}\mathbf{S}$ *if*

∂E is closed surface

(a) *S is the boundary of the solid E.*



$S = \partial E = S_1 \cup S_2$

$\operatorname{div} \vec{F} = 3x^2 + 0 + 3y^2 = 3(x^2+y^2)$

$$I = \iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \cdot dV$$

$$= 3 \iiint_E (x^2+y^2)\, dV$$

$$= 3 \iint_D \int_0^{4-x^2-y^2} (x^2+y^2)\, dz\, dA$$

$$= 3 \iint_D (4-x^2-y^2)(x^2+y^2)\, dA$$

use polar coordinates

$$= 3 \int_0^{2\pi} \int_0^2 (4-r^2) r^2 r\, dr\, d\theta$$

$$= 3 \cdot 2\pi \int_0^2 (4r^3 - r^5)\, dr$$

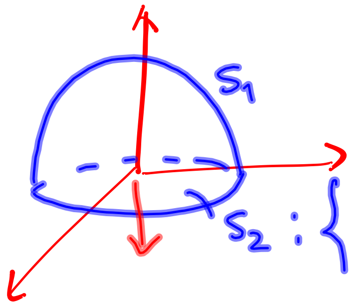
$$= 6\pi \left(r^4 - \frac{r^6}{6} \right)\Bigg|_0^2 = 6\pi \left(16 - \frac{32}{3}\right)$$

$$= 2\pi \cdot 16 = \boxed{32\pi}$$

(B) *S is the part of the paraboloid $z = 4 - x^2 - y^2$ between the planes $z = 0$ and $z = 4$.*

$S = S_1$ where S_1 is as in item (a)

Find $\displaystyle\iint_{S_1} \vec{F}\cdot d\vec{S}$



$S_2 : \{ x^2 + y^2 \le 4,\ z = 0 \}$ $\vec{n} = \langle 0, 0, -1 \rangle$

$$\underbrace{\oiint_S \vec{F}\cdot d\vec{S}}_{\substack{32\pi \\ \text{(see item (a))}}} = \iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S}$$

?

$$\iint_{S_1} \vec{F}\cdot d\vec{S} = 32\pi - \iint_{S_2} \vec{F}\cdot d\vec{S} =$$

$$= 32\pi - \iint_D \langle x^3, 0, 0 \rangle \cdot \langle 0, 0, -1 \rangle \, dA$$

$$= 32\pi - 0 = \boxed{32\pi}$$

EXAMPLE 4. *Evaluate* $I = \iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S}$ *if* S *is the boundary of*

$S = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \right\}$

(a) *ellipsoid* $E = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \right\}$ *and* $\mathbf{F} = \langle \ldots, \ldots y, \ldots \rangle$

solid

$$\oiint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \iiint_E \underbrace{\operatorname{div}(\operatorname{curl} \vec{F})}_{0} \, dV = 0$$

(b) *an arbitrary simple solid region* E *and* F *is an arbitrary continuous vector field.*

$$I = \oiint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div}(\operatorname{curl} \vec{F}) \, dV = 0$$

Note that we cannot apply here the Stokes' Theorem because surface S is closed $\Rightarrow$ no boundary curve $C = \partial S$.