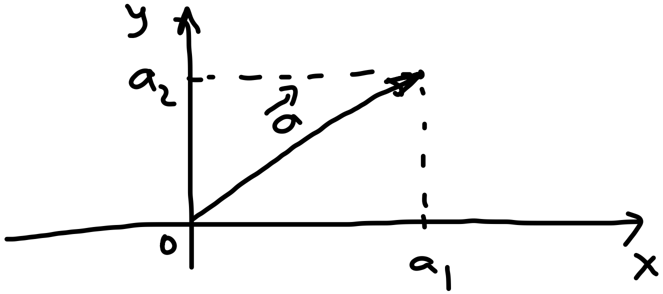
Section 1.1: Vectors

Quantities that we measure that have magnitude but not direction are called **scalars**.

DEFINITION 1. *A* ***vector*** *is a quantity that has both magnitude and direction. A 2-dimensional vector is an ordered pair* $\mathbf{a} = \langle a_1, a_2 \rangle$ *. The numbers* a_1 *and* a_2 *are called the* ***components*** *of the vector* $\mathbf{a}$.

$$\vec{a} = \langle a_1, a_2 \rangle$$



Typical notation to designate a vector is a boldfaced character or a character with and arrow on it (i.e. $\mathbf{a}$ or $\overrightarrow{a}$).

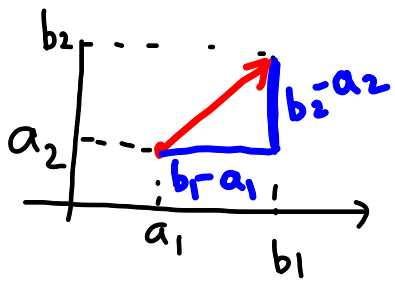
DEFINITION 2. *Given the points $A(a_1, a_2)$ and $B(b_1, b_2)$, the vector* **a** *with representation $\overrightarrow{AB}$ is*

$$\overrightarrow{AB} = \langle b_1 - a_1, b_2 - a_2 \rangle = B - A$$

The point A here is initial point and B is terminal one.

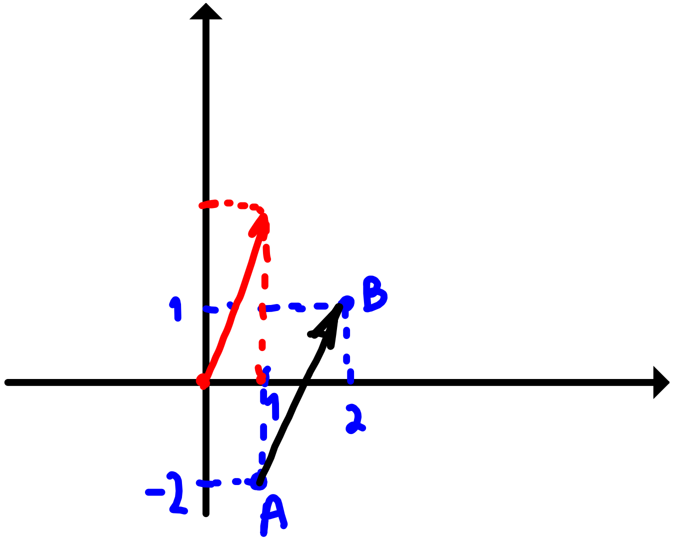
A vector with the initial point located at the origin is called the **position** vector (or we say that a vector is in standard position).

Vectors are equal if they have the same length and direction (same slope).



$$\text{slope} = \frac{b_2 - a_2}{b_1 - a_1}$$

EXAMPLE 3. *Graph the vector with initial point $A(1, -2)$ and terminal point $B(2,1)$. Find the components of $\overrightarrow{AB}$ and $\overrightarrow{BA}$.*



$$\overrightarrow{AB} = B - A = \langle 2,1\rangle - \langle 1,-2\rangle =$$

$$= \langle 2-1, 1-(-2)\rangle = \langle 1, 3\rangle$$

$$\overrightarrow{BA} = \langle 1,-2\rangle - \langle 2,1\rangle =$$

$$= \langle 1-2, -2-1\rangle = \langle -1, -3\rangle =$$

$$= -\langle 1, 3\rangle = -\overrightarrow{AB}$$

Vector operations

- *Scalar Multiplication*: If c is a scalar and $\mathbf{a} = \langle a_1, a_2 \rangle$, then

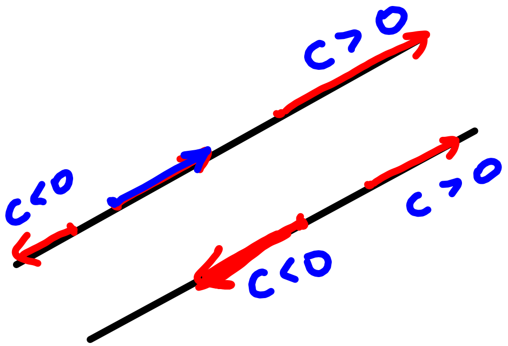
$$c\mathbf{a} = c \langle a_1, a_2 \rangle = \langle ca_1, ca_2 \rangle .$$

$2\langle 3, 4\rangle = \langle 6, 8\rangle$

$-\langle 1, 2\rangle = (-1)\cdot\langle 1, 2\rangle = \langle -1, -2\rangle$

DEFINITION 4. *Two vectors* **a** *and* **b** *are called* **parallel** *if* $\mathbf{b} = c\mathbf{a}$ *with some scalar* c.

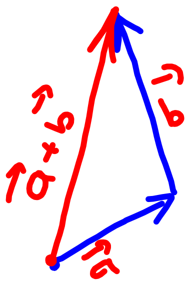
If $c > 0$ then a and ca have the same direction, if $c < 0$ then a and ca have the opposite direction.



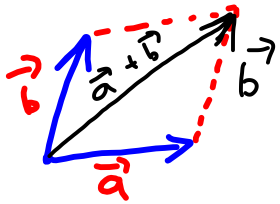
- *Vector addition*: If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ then

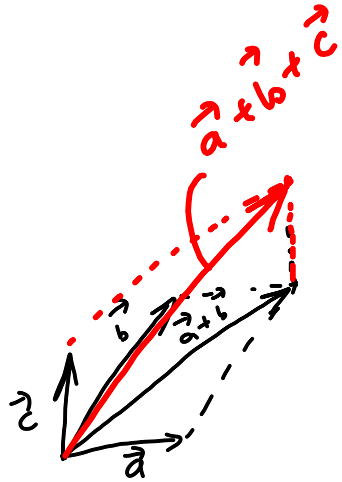
$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle .$$

Triangle Law



Parallelogram Law





$\mathbf{a} + \mathbf{b}$ is called the **resultant vector**

EXAMPLE 5. *Let* $\mathbf{a} = \langle -1, 2 \rangle$ *and* $\mathbf{b} = \langle 2.1, -0.5 \rangle$. *Then* $3\mathbf{a} + 2\mathbf{b} =$

$$3\vec{a} + 2\vec{b} = 3\langle -1, 2 \rangle + 2\langle 2.1, -0.5 \rangle =$$

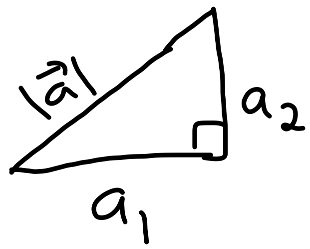
$$= \langle -3, 6 \rangle + \langle 4.2, -1 \rangle =$$

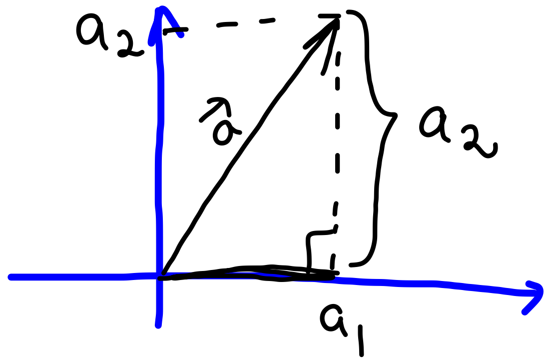
$$= \langle -3 + 4.2, 6 + (-1) \rangle =$$

$$= \langle 1.2, 5 \rangle$$

The **magnitude** or **length** of a vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is denoted by $|\mathbf{a}|$,

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$





EXAMPLE 6. *Find:* $|\langle 3, -8 \rangle|$, $\left|\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle\right|$, $|\mathbf{0}|$

$$|\langle 3, -8 \rangle| = \sqrt{3^2 + (-8)^2} = \sqrt{9 + 64} = \sqrt{73}$$

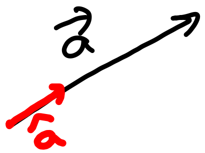
$$\left|\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle\right| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{2}{4} + \frac{2}{4}} = 1$$

$$|\vec{0}| = |\langle 0, 0 \rangle| = \sqrt{0^2 + 0^2} = 0$$

A unit vector is a vector with length one. Any vector can be made into a unit vector by dividing it by its length. So, a unit vector in the direction of **a** is

$$\hat{a} = \mathbf{u} = \frac{\mathbf{a}}{|\mathbf{a}|}.$$

normalize vector $\vec{a}$



Any vector **a** can be fully represented by providing its length, |**a**| and a unit vector **u** in its direction:

$$\mathbf{a} = |\mathbf{a}|\,\mathbf{u} = |\vec{a}| \cdot \hat{a}$$

EXAMPLE 7. *Given* $\mathbf{a} = \langle 2, -1 \rangle$. *Find*

(a) *a unit vector that has the same direction as* **a**;

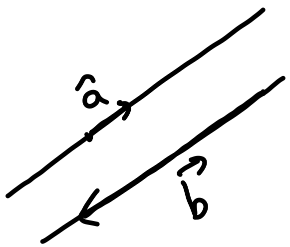
$$|\vec{a}| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\langle 2, -1 \rangle}{\sqrt{5}} = \left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right\rangle$$

(b) *a vector* **b** *in the direction opposite to* **a** *s.t* $|\mathbf{b}| = 7$.

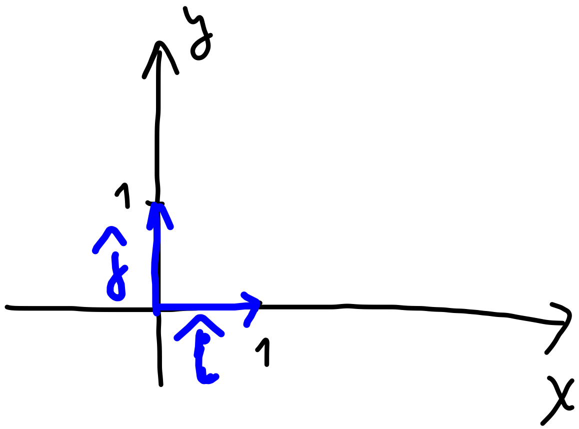
$$\vec{b} = |\vec{b}|(-\hat{a}) = -7\hat{a} = -7\left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right\rangle =$$

$$= \left\langle -\frac{14}{\sqrt{5}}, \frac{7}{\sqrt{5}} \right\rangle$$



The **standard basis vectors** are given by the unit vectors $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$ along the x and y directions, respectively. Using the basis vectors, one can represent any vector $\mathbf{a} = \langle a_1, a_2 \rangle$ as

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}.$$



$$\vec{a} = a_1\hat{i} + a_2\hat{j} =$$

$$= a_1 \langle 1, 0 \rangle + a_2 \langle 0, 1 \rangle =$$

$$= \langle a_1, 0 \rangle + \langle 0, a_2 \rangle =$$

$$= \langle a_1, a_2 \rangle$$

$$\langle 3, -4 \rangle = 3\hat{i} - 4\hat{j}$$

$$5\hat{j} = \langle 0, 5 \rangle$$

EXAMPLE 8. *Given* $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{b} = \langle 5, -2 \rangle$. *Find a scalars* s *and* t *such that* $s\mathbf{a} + t\mathbf{b} = -4\mathbf{j}$. $= -4\langle 0,1\rangle$

$$s\vec{a} + t\vec{b} = \langle 0, -4 \rangle$$

$$s\langle 2,-1\rangle + t\langle 5,-2\rangle = \langle 0,-4\rangle$$

$$\langle 2s, -s\rangle + \langle 5t, -2t\rangle = \langle 0, -4\rangle$$

$$\langle 2s+5t, -s-2t\rangle = \langle 0,-4\rangle$$

$$\begin{cases} 2s+5t=0 \\ -s-2t=-4 \end{cases} \Rightarrow s = -\frac{5}{2}t$$

$$\frac{5}{2}t - 2t = -4$$

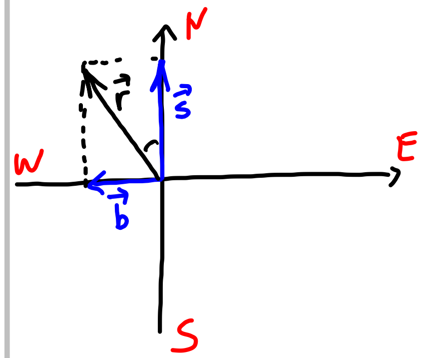
$$\frac{t}{2} = -4 \Rightarrow \boxed{t=-8}$$

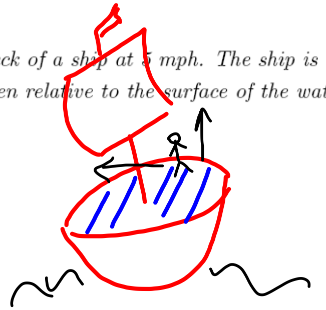
$$s = -\frac{5}{2}\cdot(-8) = 20$$

$$\boxed{t=-8,\ s=20}$$

Applications: Quantities such as force, displacement or velocity that have direction as well as magnitude are represented by vectors.

EXAMPLE 9. *Ben walks due west on the deck of a ship at 5 mph. The ship is moving north at a speed of 25 mph. Find the direction and speed of Ben relative to the surface of the water.*





Given

$|\vec{b}| = 5$

$|\vec{s}| = 25$

Find

speed $|\vec{r}| = ?$

slope of $\vec{r} = ?$

$\vec{r} = \vec{b} + \vec{s}$

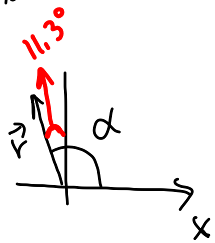
$\vec{b} = -5\hat{i}$

$s = 25\hat{j}$

$\Rightarrow \vec{r} = -5\hat{i} + 25\hat{j} = \langle -5, 25 \rangle = 5\langle -1, 5 \rangle$

speed $|\vec{r}| = 5|\langle -1, 5\rangle| = 5\sqrt{(-1)^2 + 5^2} = 5\sqrt{26}$

direction = slope $= \frac{5}{-1} = -5$



$\tan\alpha = -5$

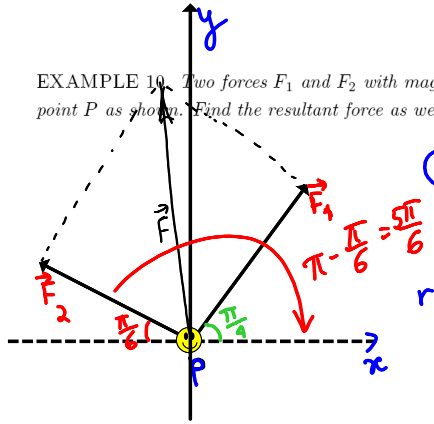
$\alpha = 101.3°$

Direction

11.3° West of North

N 11.3° W

EXAMPLE 10. *Two forces F_1 and F_2 with magnitudes 14 pounds and 12 pounds act on an object at a point P as shown. Find the resultant force as well as it's magnitude and direction.*



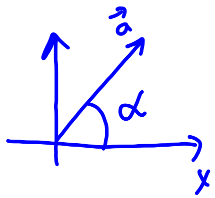
Given $|F_1| = 14$

$|F_2| = 12$

resultant $\vec{F} = \vec{F}_1 + \vec{F}_2 = ?$

$|\vec{F}| - ?$

direction - ?



$$\vec{a} = |\vec{a}|\,\hat{a} = |\vec{a}|\langle \cos\alpha, \sin\alpha \rangle$$

$$\vec{F}_1 = |\vec{F}_1|\langle \cos\frac{\pi}{4}, \sin\frac{\pi}{4} \rangle = 14\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle = \langle 7\sqrt{2}, 7\sqrt{2} \rangle$$

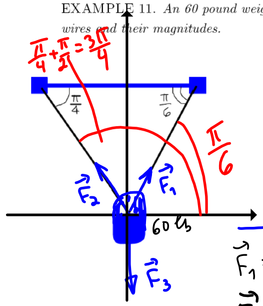
$$\vec{F}_2 = |F_2|\langle \cos\frac{5\pi}{6}, \sin\frac{5\pi}{6} \rangle = 12\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \rangle = \langle -6\sqrt{3}, 6 \rangle$$

$$\vec{F} = \vec{F}_1 + \vec{F}_2 = \langle 7\sqrt{2}, 7\sqrt{2} \rangle + \langle -6\sqrt{3}, 6 \rangle = \boxed{\langle 7\sqrt{2} - 6\sqrt{3}, 7\sqrt{2} + 6 \rangle}$$

$$|\vec{F}| = \sqrt{(7\sqrt{2} - 6\sqrt{3})^2 + (7\sqrt{2} + 6)^2} \approx 15.91 \text{ lb}$$

$$\text{slope} = \frac{7\sqrt{2} + 6}{7\sqrt{2} - 6\sqrt{3}}$$

EXAMPLE 11. *An 60 pound weight hangs from two wires as shown. Find the tensions (forces) in both wires and their magnitudes.*



Given $|\vec{F}_3| = 60$

$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \vec{0}$

Find $\vec{F}_1, \vec{F}_2, |\vec{F}_1|, |\vec{F}_2|$

$\vec{F}_1 = |\vec{F}_1|\langle \cos\frac{\pi}{6}, \sin\frac{\pi}{6}\rangle = |F_1|\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\rangle$

$\vec{F}_2 = |\vec{F}_2|\langle \cos\frac{3\pi}{4}, \sin\frac{3\pi}{4}\rangle = |F_2|\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle$

$\vec{F}_3 = |\vec{F}_3|(-\hat{j}) = 60\langle 0, -1\rangle$

$$\vec{F}_1 = \langle |F_1|\frac{\sqrt{3}}{2}, \frac{|F_1|}{2}\rangle$$
$$+\ \vec{F}_2 = \langle -|F_2|\frac{\sqrt{2}}{2}, |F_2|\frac{\sqrt{2}}{2}\rangle$$
$$\vec{F}_3 = \langle 0, -60\rangle$$

$$\vec{0} = \langle 0,0\rangle = \langle |F_1|\frac{\sqrt{3}}{2} - |F_2|\frac{\sqrt{2}}{2}, \frac{|F_1|}{2} + |F_2|\frac{\sqrt{2}}{2} - 60\rangle$$

$|\vec{F}_1| = a$, $|\vec{F}_2| = b$

$$\begin{cases} 0 = \frac{\sqrt{3}}{2}a - \frac{\sqrt{2}}{2}b \\ 0 = \frac{a}{2} + \frac{\sqrt{2}}{2}b - 60 \end{cases} \Rightarrow + \quad \begin{aligned} \sqrt{3}a - \sqrt{2}b &= 0 \\ a + \sqrt{2}b &= 120 \end{aligned}$$

$\sqrt{3}a + a = 120$

$a(\sqrt{3}+1) = 120$

$a = \frac{120(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{120(\sqrt{3}-1)}{3-1}$

$\sqrt{3}a = \sqrt{2}b \quad (\times\frac{\sqrt{2}}{2})$

$\frac{\sqrt{6}}{2}a = b$

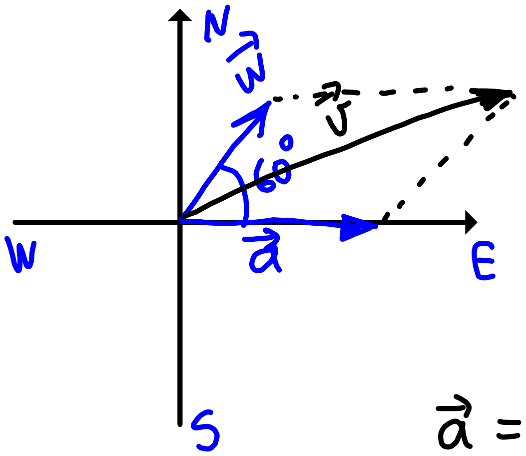
$|\vec{F}_1| = a = 60(\sqrt{3}-1)$

$b = \frac{\sqrt{6}}{2}a = \frac{\sqrt{6}}{2}\cdot 60(\sqrt{3}-1) = 30\sqrt{6}(\sqrt{3}-1) = |\vec{F}_2|$

$\vec{F}_1 = 60(\sqrt{3}-1)\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\rangle = \langle 30\sqrt{3}(\sqrt{3}-1), 30(\sqrt{3}-1)\rangle$

$\vec{F}_2 = 30\sqrt{6}(\sqrt{3}-1)\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle = \langle 30\sqrt{3}(\sqrt{3}-1), 30\sqrt{3}(\sqrt{3}-1)\rangle$

EXAMPLE 12. *An airplane, flying due East at an airspeed of 450mph, encounters a 50-mph wind acting in the direction of E60°N (60° North of East). The airplane holds its compass heading due East but, because of the wind, acquires a new ground speed (i.e. the magnitude of the resultant) and direction. What are they?*



$|\vec{a}| = 450$

$|\vec{w}| = 50$

$|\vec{v}| = ?$

slope = ?

$\vec{a} = 450\hat{\imath} = \langle 450, 0 \rangle$

$\vec{w} = 50 \langle \cos 60°, \sin 60° \rangle = 50 \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle = \langle 25, 25\sqrt{3} \rangle$

$\vec{v} = \vec{a} + \vec{w} = \langle 450 + 25, 0 + 25\sqrt{3} \rangle = \boxed{\langle 475, 25\sqrt{3} \rangle} = 25 \langle 19, \sqrt{3} \rangle$

$|\vec{v}| = 25\sqrt{19^2 + \sqrt{3}^2} = 50\sqrt{91}$

$\text{slope} = \frac{\sqrt{3}}{19} = \tan \alpha \Rightarrow \alpha \approx 5.2°$