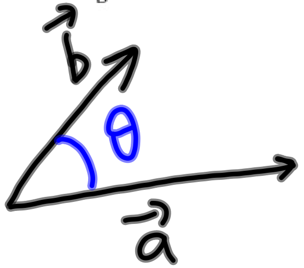
Section 1.2: The Dot Product

Let's start with two equivalent definitions of dot product.

DEFINITION 1. *The* **dot product** *of two nonzero vectors* **a** *and* **b** *is the number*

$$\boxed{\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta,}$$

where θ *is the angle between the vectors* **a** *and* **b**, $0 \le \theta \le \pi$. *If either* **a** *or* **b** *is* **0**, *then we define* $\mathbf{a}\cdot\mathbf{b} = 0$.



$$\vec{a}\cdot\vec{0} = 0$$

$$\vec{0}\cdot\vec{b} = 0$$

DEFINITION 2. *The* **dot product** *of two given vectors* $\mathbf{a} = \langle a_1, a_2\rangle$ *and* $\mathbf{b} = \langle b_1, b_2\rangle$ *is the number*

$$\boxed{\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2.}$$

Note that the formula from Definition 1 is often used not to compute a dot product but instead to find the angle between two vectors. Indeed, it implies:

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \frac{a_1b_1 + a_2b_2}{\sqrt{a_1^2+a_2^2}\,\sqrt{b_1^2+b_2^2}}$$

$$0 \le \theta \le \pi$$

EXAMPLE 3. *Given* $\mathbf{a} = \langle 2, -3 \rangle$ *and* $\mathbf{b} = \langle 3, -4 \rangle$.

(a) *Compute the dot product of* $\mathbf{a}$ *and* $\mathbf{b}$.

$$\vec{a} \cdot \vec{b} = 2 \cdot 3 + (-3) \cdot (-4) = 6 + 12 = 18$$

(b) *Determine the angle between* $\mathbf{a}$ *and* $\mathbf{b}$.

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|} = \frac{18}{\sqrt{2^2 + (-3)^2}\ \sqrt{3^2 + (-4)^2}} =$$

$$= \frac{18}{5\sqrt{13}} \Rightarrow \theta = \arccos \frac{18}{5\sqrt{13}} = \cos^{-1} \frac{18}{5\sqrt{13}}$$

Note that

$$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 \quad \Rightarrow |a| = \sqrt{a \cdot a}$$

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \quad \text{commutativity}$$

$$\vec{\mathbf{0}} \cdot \mathbf{a} = 0$$

scalar

$$\vec{a} \cdot \vec{a} = |\vec{a}| \cdot |\vec{a}| \underbrace{\cos 0}_{= 1} = |\vec{a}|^2$$

The dot product gives us a simple way for determining if two vectors are perpendicular (or orthogonal), namely,

Two nonzero vectors a and b are orthogonal if and only if a · b = 0. (*Prove it!*)

$\Rightarrow$ $\vec{a} \perp \vec{b} \Rightarrow \theta = \frac{\pi}{2} \Rightarrow \vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\underbrace{\cos\frac{\pi}{2}}_{=0} = 0$

$\Leftarrow$ $\vec{a}\cdot\vec{b} = 0 \Rightarrow \underbrace{|\vec{a}|}_{\neq 0}\cdot\underbrace{|\vec{b}|}_{\neq 0}\cdot\cos\theta = 0 \Rightarrow \cos\theta = 0 \Rightarrow$

$\Rightarrow \theta = \frac{\pi}{2} \Rightarrow \vec{a} \perp \vec{b}$

$$\boxed{\vec{a} \perp \vec{b} \iff \vec{a}\cdot\vec{b} = 0}$$

EXAMPLE 4. *Determine whether the given vectors are orthogonal, parallel, or neither.*

(a) $\langle 3,4\rangle$, $\langle -8,6\rangle$

$$\langle 3,4\rangle\cdot\langle -8,6\rangle = 3\cdot(-8)+4\cdot 6 = -24+24=0$$

$$\Rightarrow \langle 3,4\rangle \perp \langle -8,6\rangle$$

(b) $\langle -7,-4\rangle$, $\langle 28,16\rangle$

$$\langle -7,-4\rangle\cdot\langle 28,16\rangle = -7\cdot 28+(-4)\cdot 16 \neq 0$$

Note $\langle 28,16\rangle = -4\langle -7,-4\rangle \Rightarrow \langle -7,-4\rangle \parallel \langle 28,16\rangle$

$$\theta = \pi$$

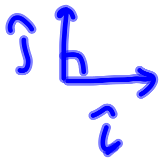
(c) $\langle 1,1\rangle$, $\langle 2,3\rangle$

$$\cos\theta = \frac{\langle 1,1\rangle\cdot\langle 2,3\rangle}{|\langle 1,1\rangle|\cdot|\langle 2,3\rangle|} = \frac{2+3}{\sqrt{2}\sqrt{13}} = \frac{5}{\sqrt{26}}$$

neither, $\theta = \arccos\frac{5}{\sqrt{26}} = \ldots$

EXAMPLE 5. *What is the dot product of* $12\mathbf{j}$ *and* $11\mathbf{i}$*?*

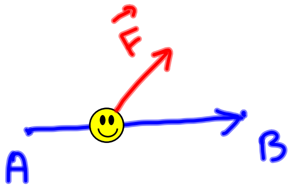
$$\underbrace{12\hat{j}\cdot 11\hat{i}}_{=\,0} = \underbrace{\langle 0, 12\rangle \cdot \langle 11, 0\rangle = 0.}_{\text{unnecessarily}}$$



DEFINITION 6. *The* **work** *done by a force* $\mathbf{F}$ *in moving and object from point* A *to point* B *is given by*

$$\boxed{W = \mathbf{F} \cdot \mathbf{D}}$$

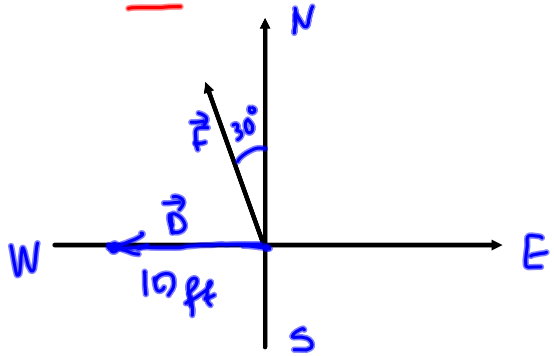
where $\mathbf{D} = \overrightarrow{AB}$ *is the distance the object has moved (or displacement).*



EXAMPLE 7. *Find the work done by a force of* 50*lb acting in the direction* $N30^oW$ *in moving an object* 10ft *due west.*

$|\vec{F}| = 50$

$|\vec{D}| = 10$

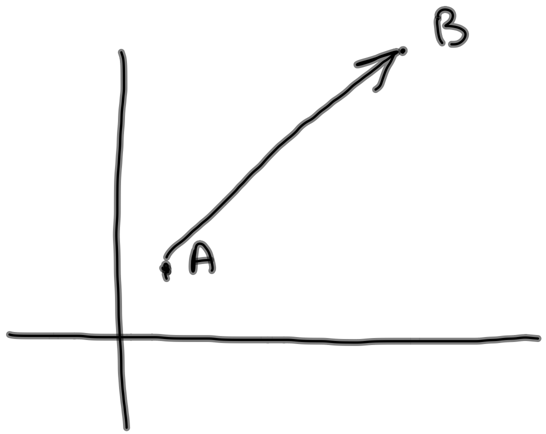


$$W = \vec{F} \cdot \vec{D} = |\vec{F}| \cdot |\vec{D}| \cdot \cos\theta$$

where $\theta = \measuredangle \vec{F}, \vec{D} = 90^\circ - 30^\circ = 60^\circ$

$$W = 50 \cdot 10 \cos 60^\circ = \frac{500}{2} = 250 \text{ ft-lb}$$

EXAMPLE 8. *A constant force* $\mathbf{F} = 25\mathbf{i} + 4\mathbf{j}$ *(the magnitude of* $\mathbf{F}$ *is measured in Newtons) is used to move an object from* $A(1,1)$ *to* $B(5,6)$. *Find the work done if the distance is measured in meters*



$W = \vec{F} \cdot \vec{D}$

$\vec{D} = \overrightarrow{AB} = \langle 5,6 \rangle - \langle 1,1 \rangle = \langle 4,5 \rangle$

$W = \langle 25,4 \rangle \cdot \langle 4,5 \rangle =$

$= 25 \cdot 4 + 4 \cdot 5 = 120 \text{ J}$

DEFINITION 9. *The* **orthogonal compliment of a** $= \langle a_1, a_2\rangle$ **is** $\boxed{\mathbf{a}^{\perp} = \langle -a_2, a_1\rangle}$

Note that $|\mathbf{a}| = |\mathbf{a}^{\perp}|$ and $\mathbf{a}\cdot\mathbf{a}^{\perp} = \langle a_1, a_2\rangle \cdot \langle -a_2, a_1\rangle = -a_1 a_2 + a_2 a_1 = 0 \Rightarrow$

$$\sqrt{a_1^2+a_2^2} = \sqrt{(-a_2)^2+a_1^2} \qquad \Rightarrow \vec{a} \perp a^{\perp}$$

parallel

EXAMPLE 10. *Given* $\langle 4, -2\rangle$, $\langle 2, -1\rangle$, $\underbrace{\langle -2, 1\rangle}$ *and* $\mathbf{a} = \langle 1, 2\rangle$. *Which of these vectors is*

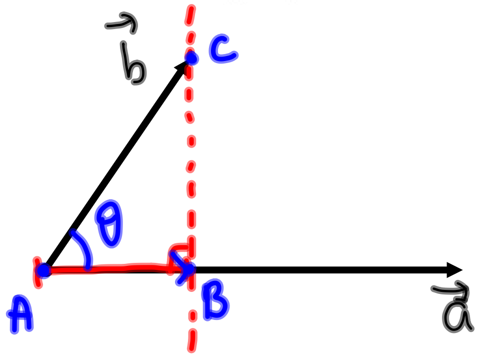
- *orthogonal to* **a**?

  $\langle -2, 1\rangle \cdot \langle 1, 2\rangle = 0$

- *the orthogonal compliment of* **a**?

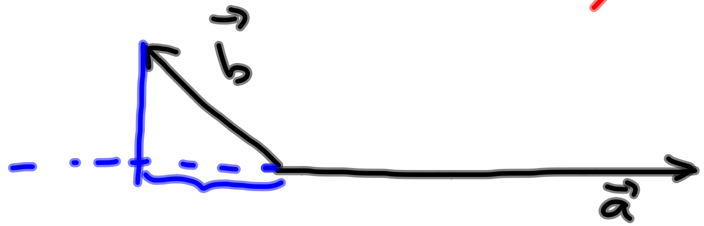
  $\langle -2, 1\rangle = \vec{a}^{\perp}$

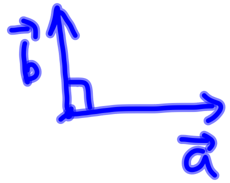
Scalar and vector projections: For given two vectors **a** and **b** we determine the projection of **b** onto **a**.



$|\vec{AB}| = ?$

$|\vec{AB}| = |\vec{b}| \cdot \cos\theta = \cancel{|\vec{b}|} \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}| \cdot \cancel{|\vec{b}|}}$





- The vector projection of **b** onto **a** is denotes by $\text{proj}_{\mathbf{a}}\mathbf{b}$ and can be found by the formula

$$\text{a vector} = \text{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\,\mathbf{a}.$$

- The scalar projection of **b** onto **a** (or the component of **b** along **a**) is denotes by $\text{comp}_{\mathbf{a}}\mathbf{b}$ and can be found by the formula

$$\text{a scalar} \leftarrow \underbrace{\text{comp}_{\mathbf{a}}\mathbf{b}} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}.$$

$$\text{proj}_{\vec{a}}\vec{b} = \text{comp}_{\vec{a}}\vec{b} \cdot \hat{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} \cdot \frac{\vec{a}}{|\vec{a}|}$$

EXAMPLE 11. *Given* $\mathbf{a} = \langle 4, 3\rangle$ *and* $\mathbf{b} = \langle 1, -1\rangle$. *Find:*

- $\mathbf{a}\cdot\mathbf{b} =$ $4\cdot 1 + 3\cdot(-1) = 1 = \vec{b}\cdot\vec{a}$
- $|\mathbf{a}| =$ $\sqrt{4^2+3^2} = 5$
- $|\mathbf{b}| =$ $\sqrt{1^2+(-1)^2} = \sqrt{2}$
- $\mathrm{proj}_{\mathbf{b}}\mathbf{a} =$ $\dfrac{\vec{b}\cdot\vec{a}}{|\vec{b}|^2}\cdot\vec{b} = \dfrac{1}{(\sqrt{2})^2}\langle 1,-1\rangle = \frac{1}{2}\langle 1,-1\rangle = \langle \frac{1}{2}, -\frac{1}{2}\rangle$
- $\mathrm{comp}_{\mathbf{a}}\mathbf{b} =$ $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \dfrac{1}{5}$

$\mathrm{comp}_{3\vec{a}}\vec{b} = \frac{1}{5}$ $\qquad$ $\mathrm{comp}_{\vec{a}}(7\vec{b}) = \frac{7}{5}$

$\mathrm{comp}_{5\vec{a}}\vec{b} = \frac{1}{5}$ $\qquad$ $\mathrm{com}_{4\vec{a}}(3\vec{b}) = \frac{3}{5}$

Note $\mathrm{comp}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \dfrac{\vec{a}}{|\vec{a}|}\cdot\vec{b} = \hat{a}\cdot\vec{b}$

$k>0$ $\quad$ $\mathrm{comp}_{k\vec{a}}\vec{b} = \hat{a}\cdot\vec{b}$

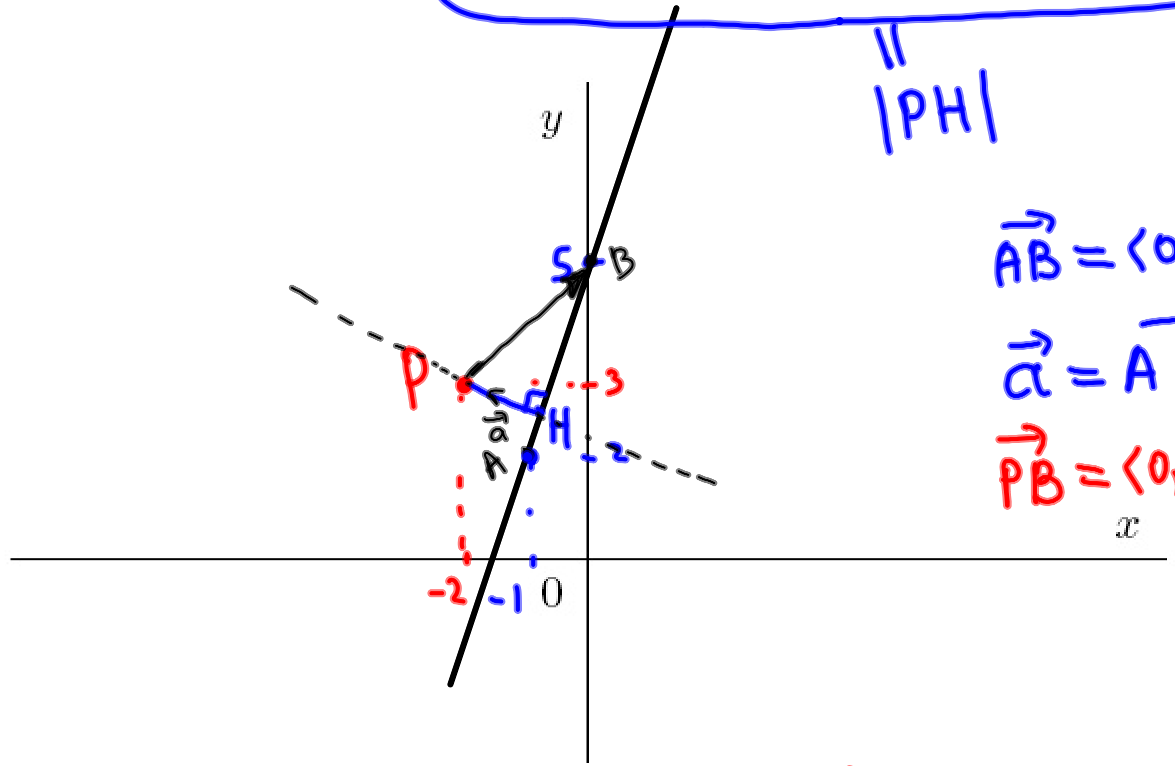
$\mathrm{comp}_{\vec{a}}(k\vec{b}) = \hat{a}\cdot(k\vec{b}) = k(\hat{a}\cdot\vec{b}) = k\,\mathrm{comp}_{\vec{a}}\vec{b}$

$\mathrm{comp}_{\vec{a}}\vec{b} = \hat{a}\cdot\vec{b}$ $\qquad$ $\mathrm{proj}_{\vec{a}}\vec{b} = (\hat{a}\cdot\vec{b})\,\hat{a}$

EXAMPLE 12. *Find the distance from the point $P(-2,3)$ to the line $y=3x+5$.*

$= |PH|$

x	0	-1
y	5	2



$\overrightarrow{AB} = \langle 0,5\rangle - \langle -1,2\rangle = \langle 1,3\rangle$

$\vec{a} = \overrightarrow{AB}^{\perp} = \langle -3,1\rangle$

$\overrightarrow{PB} = \langle 0,5\rangle - \langle -2,3\rangle =$

$= \langle 2,2\rangle$

$$|PH| = \left|\operatorname{comp}_{\vec{a}} \overrightarrow{PB}\right| = \left|\frac{\vec{a}\cdot\overrightarrow{PB}}{|\vec{a}|}\right| = \left|\frac{\langle -3,1\rangle\cdot\langle 2,2\rangle}{\sqrt{(-3)^2+1^2}}\right| =$$

$$= \left|\frac{-6+2}{\sqrt{10}}\right| = \frac{4}{\sqrt{10}}$$