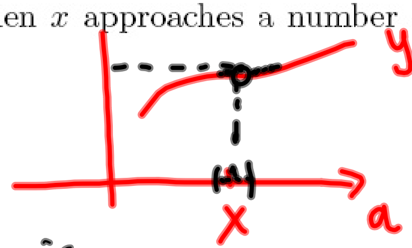
Section 2.2: The Limit of a function

$$f(-x) = \frac{\sin(-x)}{-x} = \frac{-\sin x}{-x} = \frac{\sin x}{x} = f(x)$$

A limit is a way to discuss how the values of a function $f(x)$ behave when x approaches a number a, whether or not $f(a)$ is defined.



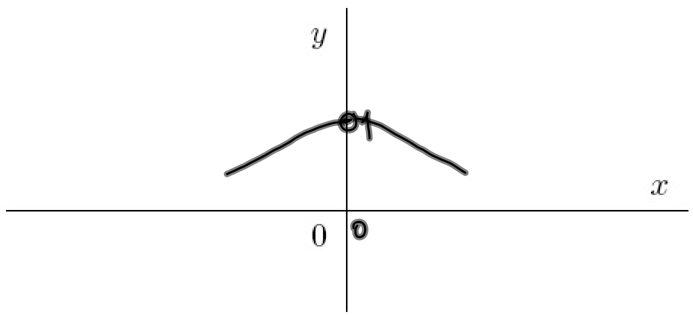
Let's consider the following function:

even $\quad f(x) = \dfrac{\sin x}{x} \qquad (x \text{ in radians}).$

⇒ symmetric w.r.t. the y-axis

Note that $f(0) = \dfrac{\sin 0}{0}$ is undefined. However, one can compute the values of $f(x)$ for values of x close to 0.

x	$f(x)$
$\pm$ 0.1	0.99833417
$\pm$ 0.05	0.99958339
$\pm$ 0.01	0.99998333
$\pm$ 0.005	0.99999583
$\pm$ 0.001	0.99999983



The table allows us to guess (correctly) that that our function gets closer and closer to 1 as x approaches 0 through positive and negative values. In limit notation it can be written as

Left hand limit $\displaystyle \lim_{x\to 0^-} \frac{\sin x}{x} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1$ Right hand limit

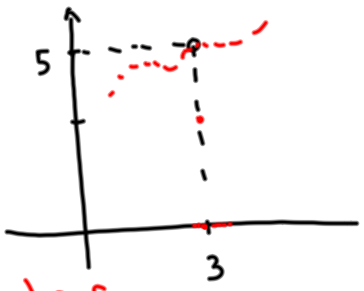
which implies that

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1.}$$ FACT

DEFINITION 1.

- *If* $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$ *then* $\lim_{x\to a} f(x)$ *exists and* $\boxed{\lim_{x\to a} f(x) = L}$*;*
- *If* $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x) = L$ *then* $\lim_{x\to a} f(x)$ *does not exist.*

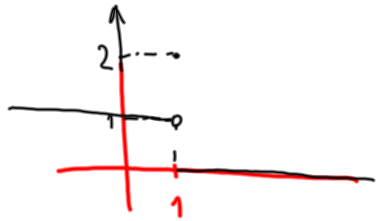
$$\lim_{x\to 3} f(x) = 5$$



$f(2.9999\ldots) \approx 5$

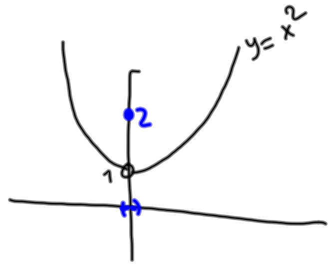
$f(3{,}000001) \approx 5$

$f(3) = ?$



$$f(x) = \begin{cases} 1, & x < 1 \\ 2, & x = 1 \\ 0, & x > 1 \end{cases}$$

$\lim_{x\to 1} f(x)$ DNE

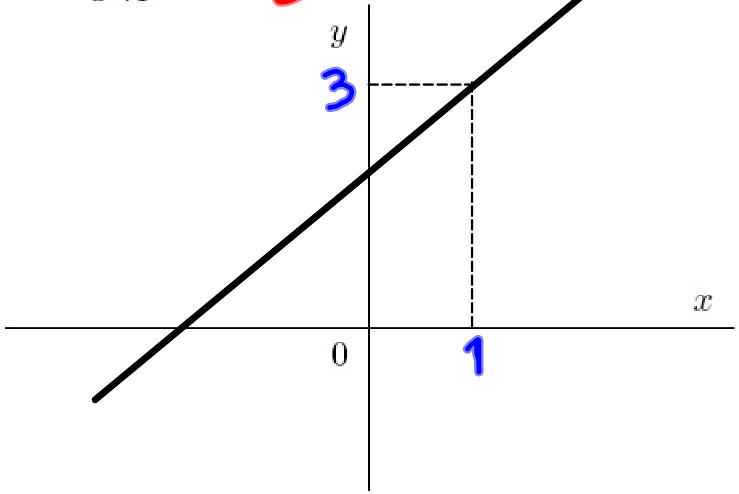


$$f(x) = \begin{cases} x^2+1, & x \neq 0 \\ 2, & x = 0 \end{cases}$$

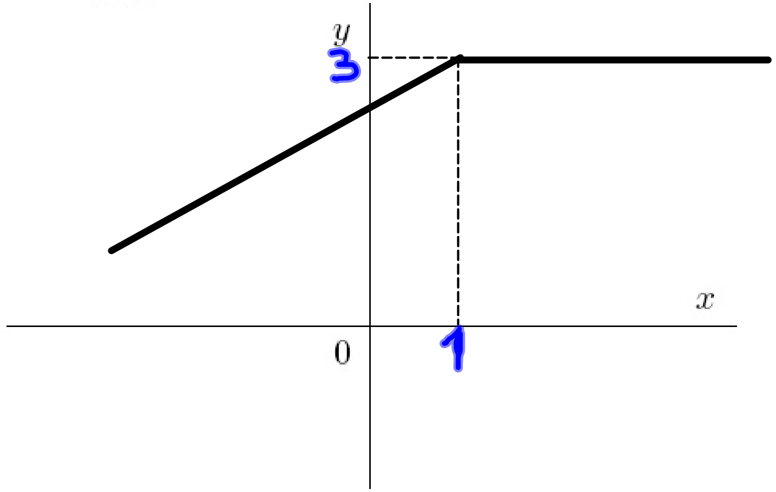
$\lim_{x\to 0} f(x) = 1$.

However $f(0) = 2 \neq 1$

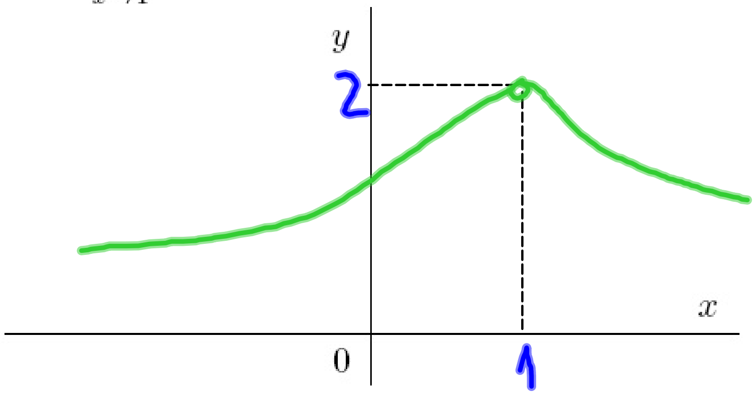
$$\lim_{x \to 1} f(x) = 3$$



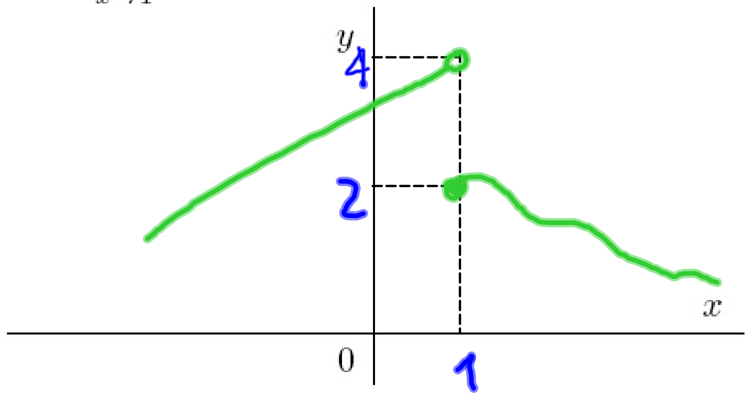
$$\lim_{x \to 1} f(x) = 3$$



$\lim_{x\to 1} f(x) = 2$



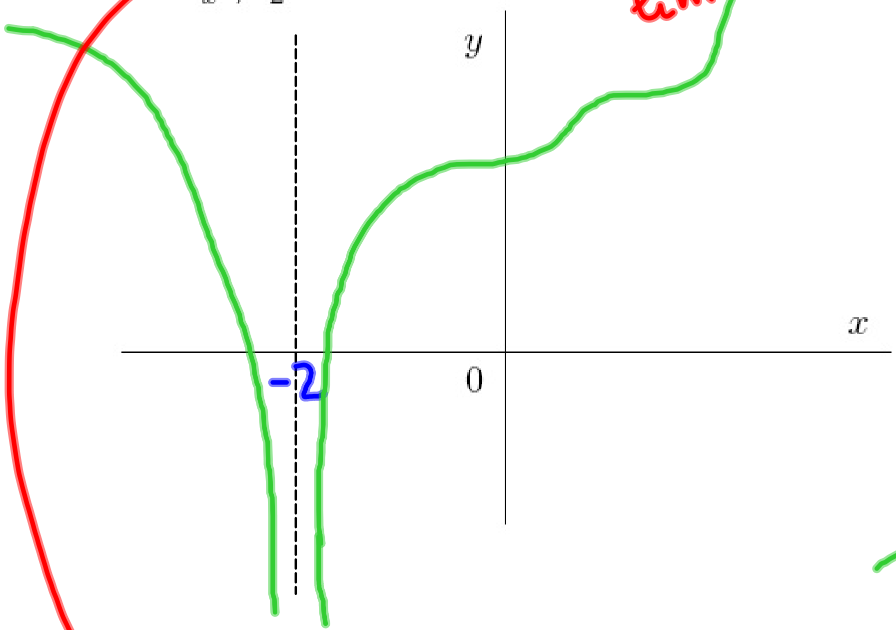
$\lim_{x\to 1} f(x) =$ DNE



$$\lim_{x\to 1^-} f(x) = 4$$

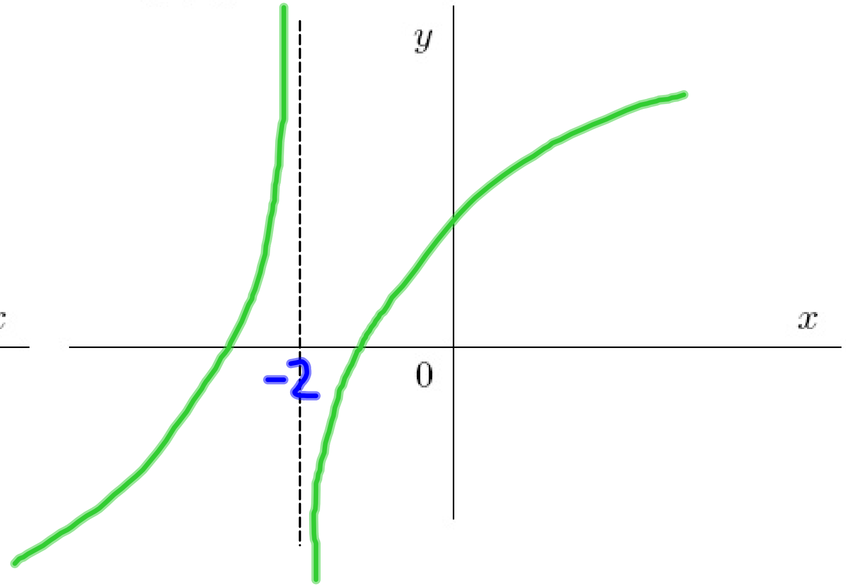
$$\lim_{x\to 1^+} f(x) = 2$$

$\lim_{x \to -2} f(x) = -\infty$ as infinite limit



DNE as a finite limit

$\lim_{x \to -2} f(x) =$ DNE



$\lim_{x \to -2^-} f(x) = \infty$

$\lim_{x \to -2^+} f(x) = -\infty$

Limits of piecewise defined function.

EXAMPLE 2. *Plot the graph of the function*

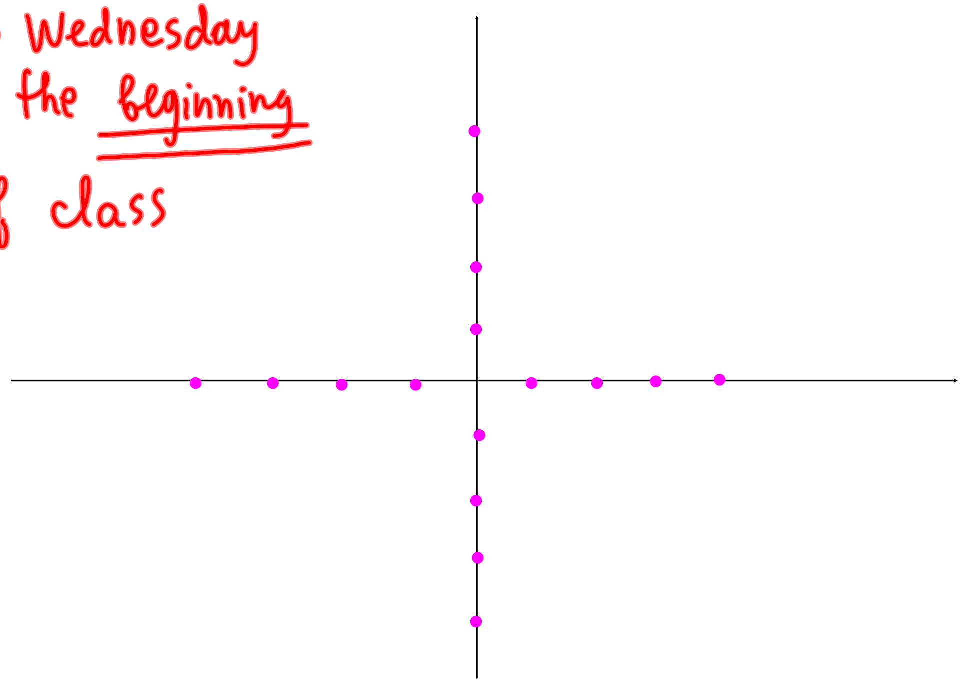
$$f(x) = \begin{cases} -3 - x & \text{if} \quad x \le -2 \\ 2x & \text{if} \quad -2 < x < 2 \\ x^2 - 4x + 3 & \text{if} \quad x \ge 2 \end{cases}$$

x	y

Name ______________

Due Wednesday at the beginning of class

x	y



x	y

Find the limits (using the graph above):

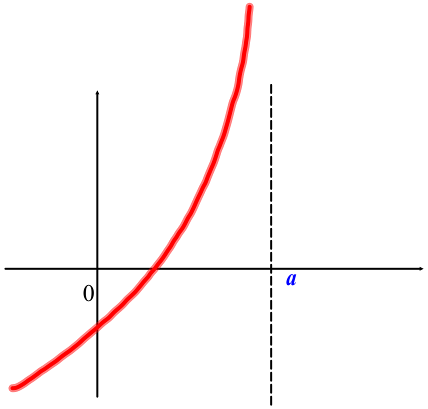
$\lim\limits_{x\to 0^-} f(x) =$ $\qquad$ $\lim\limits_{x\to 0^+} f(x) =$ $\qquad$ $\lim\limits_{x\to 0} f(x) =$

$\lim\limits_{x\to -2^-} f(x) =$ $\qquad$ $\lim\limits_{x\to -2^+} f(x) =$ $\qquad$ $\lim\limits_{x\to -2} f(x) =$

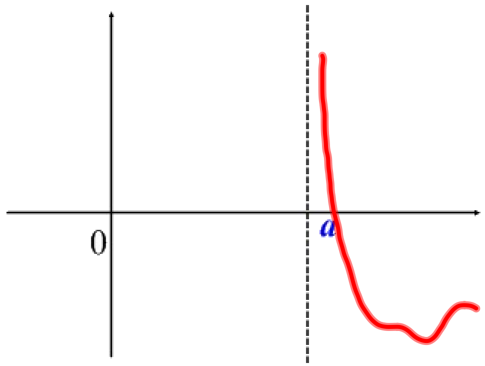
$\lim\limits_{x\to 2^-} f(x) =$ $\qquad$ $\lim\limits_{x\to 2^+} f(x) =$ $\qquad$ $\lim\limits_{x\to 2} f(x) =$

DEFINITION 3. *The line $x = a$ is said to be a* **vertical asymptote** *of the curve $y = f(x)$ if at least one of the following six statements is true:*

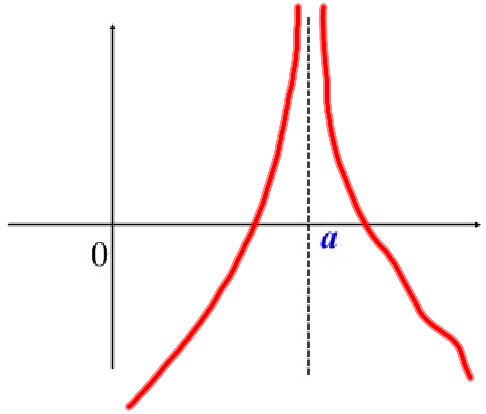
$$\lim_{x \to a^-} f(x) = \infty$$



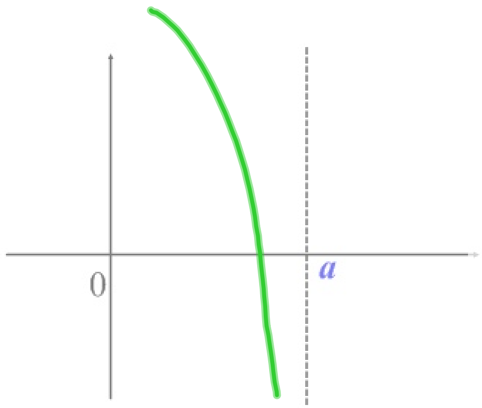
$$\lim_{x \to a^+} f(x) = \infty$$



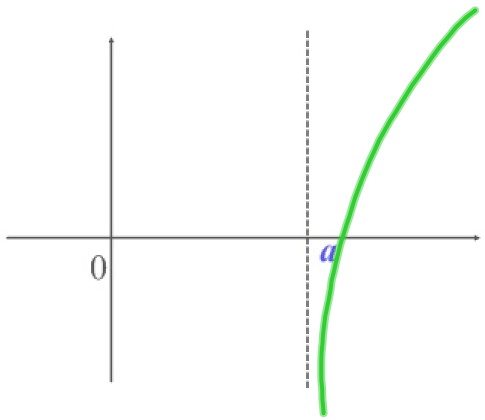
$$\lim_{x \to a} f(x) = \infty$$



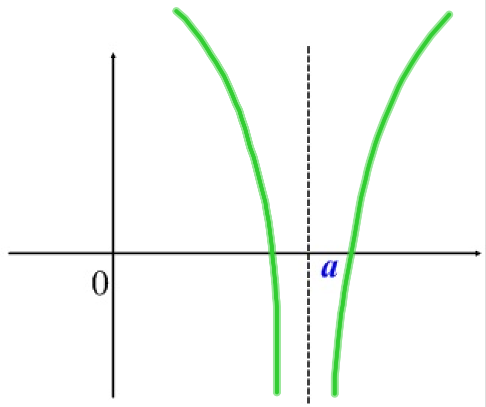
$$\lim_{x \to a^-} f(x) = -\infty$$



$$\lim_{x \to a^+} f(x) = -\infty$$



$$\lim_{x \to a} f(x) = -\infty$$



REMARK 4. The vertical asymptotes of a rational function come from the zeroes of the denominator.

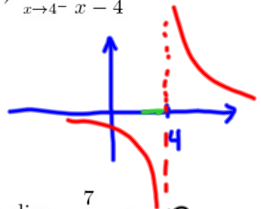
EXAMPLE 5. *Determine the infinite limit:*

(a) $\lim\limits_{x\to 4^-} \frac{7}{x-4} = -\infty$

$y = \frac{7}{x-4}$ is rational

$x < 4 \Rightarrow x - 4 < 0 \Rightarrow$

$\Rightarrow \frac{7}{x-4} < 0 \Rightarrow -\infty$



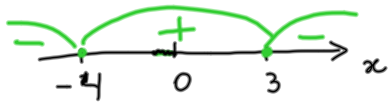
(b) $\lim\limits_{x\to 4^+} \frac{7}{x-4} = \infty$

$x > 4 \Rightarrow \frac{7}{x-4} > 0$

(c) $\lim\limits_{x\to 4} \frac{7}{x-4} =$ DNE (see (a) & (b))

(d) $\lim\limits_{x\to 0^-} \frac{3-x}{x^4(x+4)} = \infty$

$x < 0$

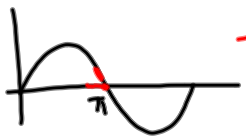


(e) $\lim\limits_{x\to 0^+} \frac{3-x}{x^4(x+4)} = \infty$

(f) $\lim\limits_{x\to 0} \frac{3-x}{x^4(x+4)} = \infty$

(g) $\lim\limits_{x\to \pi^-} \csc x = \lim\limits_{x\to\pi^-} \frac{1}{\sin x} = \infty$

Determine sign of $y = \sin x$ when $x < \pi$ (closed to π)



$\rightarrow \sin x > 0 \Rightarrow \frac{1}{\sin x} > 0$

EXAMPLE 6. *Given:* $f(x) = \dfrac{x-4}{x^2-5x+4}$.

(a) *What are the vertical asymptotes of* $f(x)$*?*

Find zeroes of denominator

$x^2 - 5x + 4 = 0$

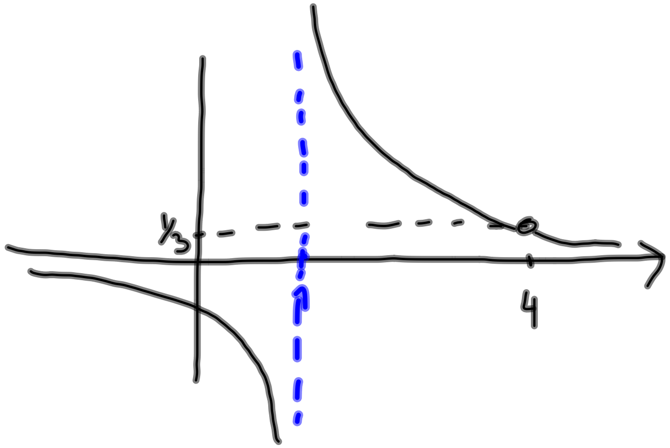
$(x-4)(x-1) = 0$

$$f(x) = \frac{\cancel{x-4}}{\cancel{(x-4)}(x-1)} = \frac{1}{x-1}$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{1}{x-1} = -\infty$$

$\Rightarrow x = 1$ is vert. asymptote.

(b) *How does* $f(x)$ *behave near the asymptotes?*



$$\lim_{x \to 4} f(x) = \lim_{x \to 4} \frac{1}{x-1} = \frac{1}{4-1} = \frac{1}{3}$$

Note $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{1}{x-1} = \infty$