## Section 2.3: Calculating limits using the limits laws

LIMIT LAWS Suppose that $c$ is a constant and the limits

as finite number

$$\lim_{x\to a} f(x) \qquad \text{and} \qquad \lim_{x\to a} g(x)$$

exist. Then

Also hold for vector-functions

1. $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
2. $\lim_{x\to a} [cf(x)] = c \lim_{x\to a} f(x)$
3. $\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$
4. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$
5. $\lim_{x\to a} c = c$
6. $\lim_{x\to a} x = a$ ← by direct substitution
7. $\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$, where $n$ is a positive integer.
8. $\lim_{x\to a} x^n = a^n$, where $n$ is a positive integer.
9. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where $n$ is a positive integer and if $n$ is even, then we assume that $\lim_{x\to a} f(x) > 0$.
10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{\lim_{x\to a} x}$ where $n$ is a positive integer and if $n$ is even, then we assume that $a > 0$.

REMARK 1. Note that *all these properties also hold for the one-sided limits.*

REMARK 2. The analogues of the *laws 1-3 also hold when $f$ and $g$ are vector functions* (the product in Law 3 should be interpreted as a dot product).

EXAMPLE 3. *Compute the limit:*
$\lim_{x\to -1}(7x^5+2x^3-8x^2+3) =$

$7\lim_{x\to -1} x^5 + 2\lim_{x\to -1} x^3 - 8\lim_{x\to -1} x^2 + \lim_{x\to -1} 3 =$

(8) (5) (1) (2)

(8)+(5) $7\cdot(-1)^5 + 2(-1)^3 - 8(-1)^2 + 3 = -14$

REMARK 4. If we had defined $f(x) = 7x^5+2x^3-8x^2+3$ then Example 3 would have been,

$$\lim_{x\to -1} f(x) = \lim_{x\to -1}(7x^5+2x^3-8x^2+3) = 7(-1)^5+2(-1)^3-8(-1)^2+3 = -14 = f(-1)$$

Since x=-1 belongs to the domain of f(x), we can use DIRECT SUBSTITUTION RULE to calculate the limit at that point.

EXAMPLE 5. *Compute the limit:*
$\lim_{x\to -2}\frac{x^2+x+1}{x^3-10} =$ $\frac{(-2)^2+(-2)+1}{(-2)^3-10} = -\frac{1}{6}$

$x=-2$ is in the domain of $f(x) = \frac{x^2+x+1}{x^3-10}$ => we can use

REMARK 6. The function from Example 5 also satisfies "direct substitution property":

$$\lim_{x\to a} f(x) = f(a).$$

Later we will say that such functions are *continuous*. Note that in both examples it was important that $a$ in the domain of $f$.

For example, the function $f(x) = \frac{x^2+x+1}{x^3-10}$ is continuous at $x=-2$ because

$$\lim_{x\to -2} f(x) = f(-2).$$

Moreover, $f(x)$ is continuous for all $x$ such that $x \neq \sqrt[3]{10}$

EXAMPLE 7. *Compute the limit:*

$$\lim_{x\to 3}\frac{x-3}{x^2-9} = \lim_{x\to 3}\frac{\cancel{x-3}}{\cancel{(x-3)}(x+3)} = \lim_{x\to 3}\frac{1}{x+3} = \frac{1}{3+3} = \frac{1}{6}$$

EXAMPLE 8. *Compute the limit:*

$$\lim_{x\to 1}\frac{x-1}{x^2-4x+3} = \lim_{x\to 1}\frac{\cancel{x-1}}{\cancel{(x-1)}(x-3)} = \lim_{x\to 1}\frac{1}{x-3} = \frac{1}{1-3} = -\frac{1}{2}$$

We used factor

EXAMPLE 9. *Given*

$$g(x) = \begin{cases} x^2 + 4, & \text{if} \quad x \le -1 \quad \text{left} \\ 2 - 3x & \text{if} \quad x > -1 \quad \text{right} \end{cases}$$

*Compute the limits:*

(a) $\lim_{x \to 4} g(x) = g(4) = 2 - 3 \cdot (4) = -10$

(b) $\lim_{x \to -1} g(x) = 5$

Use left hand and right hand limits

$$\lim_{x \to -1^-} g(x) = \lim_{x \to -1^-} (x^2 + 4) = (-1)^2 + 4 = 5$$

$$\lim_{x \to -1^+} g(x) = \lim_{x \to -1^+} (2 - 3x) = 2 - 3 \cdot (-1) = 5$$

EXAMPLE 10. *Evaluate these limits.*

(a) $\lim\limits_{x\to 4} \dfrac{x^{-1}-0.25}{x-4} = \lim\limits_{x\to 4} \dfrac{\frac{1}{x}-\frac{1}{4}}{x-4} = \lim\limits_{x\to 4} \dfrac{4-x}{4x(x-4)} =$

$= \lim\limits_{x\to 4} -\dfrac{\cancel{x-4}}{4x\cancel{(x-4)}} = -\dfrac{1}{4}\lim\limits_{x\to 4}\dfrac{1}{x} = -\dfrac{1}{4}\cdot\dfrac{1}{4} = \boxed{-\dfrac{1}{16}}$

(b) $\lim\limits_{x\to 0} \dfrac{(x+5)^2-25}{x} = \lim\limits_{x\to 0} \dfrac{(x+5)^2-5^2}{x} = \lim\limits_{x\to 0} \dfrac{(\cancel{x+5-5})(x+5+5)}{\cancel{x}}$

$= \lim\limits_{x\to 0} (x+10) = \boxed{10}$

(c) $\lim_{x\to -1} \frac{|x+1|}{x+1}$ DNE

$$f(x) = \frac{|x+1|}{x+1} = \begin{cases} \frac{x+1}{\cancel{x+1}} = 1, & x+1 \geq 0 \Rightarrow x \geq -1 \quad \text{Right} \\ \frac{-(x+1)}{\cancel{x+1}} = -1 & x+1 \leq 0 \Rightarrow x \leq -1 \quad \text{Left} \end{cases}$$


$$1 = \lim_{x\to -1^+} f(x) \neq \lim_{x\to -1^-} f(x) = -1$$

(d) $\lim_{x\to -1} \frac{x^2+x}{|x+1|} = \lim_{x\to -1} \frac{x(x+1)}{|x+1|}$

$$f(x) = \frac{x(x+1)}{|x+1|} = \begin{cases} \frac{x\cancel{(x+1)}}{\cancel{x+1}} = x, & x \geq -1 \\ \frac{x\cancel{(x+1)}}{-\cancel{(x+1)}} = -x & x \leq -1 \end{cases}$$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} x = -1$$

$$\neq \Rightarrow \lim_{x\to -1} f(x) \text{ DNE}$$

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (-x) = 1$$

(e) $\lim_{x\to 0^-} \left\{ \frac{1}{x} - \frac{1}{|x|} \right\} = \lim_{x\to 0^-} \frac{2}{x} = -\infty$ ( DNE as a finite limite)

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x \leq 0 \end{cases}$$

$$f(x) = \frac{1}{x} - \frac{1}{|x|} = \begin{cases} \frac{1}{x} - \frac{1}{x} = 0, & x \geq 0 \\ \frac{1}{x} - \left(-\frac{1}{x}\right) = \frac{2}{x}, & x \leq 0 \quad \text{left} \end{cases}$$

(f) $\lim_{x\to 0} \frac{\sqrt{6-x} - \sqrt{6}}{x} \frac{\left(\sqrt{6-x} + \sqrt{6}\right)}{\sqrt{6-x} + \sqrt{6}} = \lim_{x\to 0} \frac{\cancel{6} - \overset{-1}{\cancel{x}} - \cancel{6}}{\cancel{x}\left(\sqrt{6-x} + \sqrt{6}\right)} = -\lim_{x\to 0} \frac{1}{\sqrt{6-x}+\sqrt{6}} = \boxed{-\frac{1}{2\sqrt{6}}}$

$(a-b)(a+b) = a^2 - b^2$

$(\sqrt{c} - \sqrt{d})(\sqrt{c} + \sqrt{d}) = c - d$

**Conclusion** from the above examples:

To calculate the limit of $f(x)$ as $x \to a$:

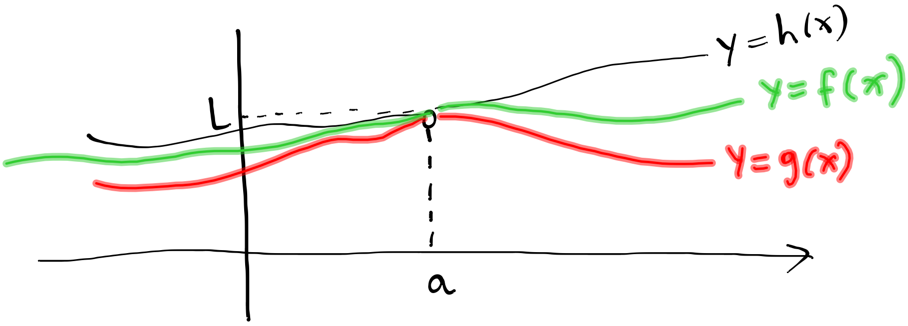
- PLUG IN $x = a$ if $a$ is in the domain of $f$.
- Otherwise "FACTOR" or "MULTIPLY BY CONJUGATE" and then plug in.
- Consider one sided limits if necessary.
- Squeeze Theorem

**Squeeze Theorem.** Suppose that for all $x$ in an interval containing $a$ (except possibly at $x = a$)

$$\boxed{g(x) \le f(x) \le h(x)}$$

and $\lim_{x\to a} g(x) = L = \lim_{x\to a} h(x)$. Then

$$\lim_{x\to a} f(x) = L.$$




$$\begin{array}{ccccc} g(x) & \le & f(x) & \le & h(x) \\ \downarrow & & | & & \downarrow \\ L & & & & L \end{array}$$

$$\downarrow$$

$$L$$

**Corollary.** Suppose that for all $x$ in an interval containing $a$ (except possibly at $x = a$)

$$|f(x)| \le h(x) \quad (\textit{equivalently}, \quad -h(x) \le f(x) \le h(x))$$

and $\lim_{x\to a} h(x) = 0$. Then

$$\lim_{x\to a} f(x) = 0.$$

$$\lim_{x\to a}\big(-h(x)\big) = -\lim_{x\to a} h(x) = 0$$

$$-h(x) \le f(x) \le h(x)$$

EXAMPLE 11. *Given* $3x \le f(x) \le x^3 + 2$ *for* $0 \le x \le 2$. *Find* $\lim_{x\to 1} f(x)$ $= 3$

$$3x \xrightarrow{x\to 1} 3, \qquad f(x) \xrightarrow{x\to 1} 3, \qquad x^3 + 2 \xrightarrow{x\to 1} 3$$

EXAMPLE 12. *Evaluate:*

(a) $\lim_{x\to 0} x \sin\frac{1}{x} = 0$

$$\left|x \sin\frac{1}{x}\right| = |x| \cdot \underbrace{\left|\sin\frac{1}{x}\right|}_{\leq 1} \leq |x| \xrightarrow[x\to 0]{} 0$$

(b) $\lim_{t\to 0}(t^5)\cos^3(\frac{1}{t^2}) = 0$

$$\left|t^5 \cos^3\frac{1}{t^2}\right| = |t|^5 \cdot \left|\cos\frac{1}{t^2}\right|^3 \leq |t|^5 \cdot 1^3 = |t| \xrightarrow[t\to 0]{} 0$$

EXAMPLE 13. *Is there a number $c$ such that*

$$\lim_{x \to -2} \frac{3x^2 + cx + c + 3}{x^2 + x - 2} = \lim_{x \to -2} \frac{3x^2 + cx + c + 3}{(x+2)(x-1)}$$

*exists? If so, find the value $c$ and the value of the limit.*

Such $c$ does exist if $x = -2$ is a zero of the numerator, i.e.

$$3x^2 + cx + c + 3 = 0 \quad \text{for } x = -2$$

$$3(-2)^2 + c\cdot(-2) + c + 3 = 0$$

$$12 - 2c + c + 3 = 0$$

$$\boxed{c = 15}$$

Find the limit:

$$\lim_{x \to -2} \frac{3x^2 + 15x + \overbrace{15 + 3}^{18}}{(x+2)(x-1)} = \lim_{x \to -2} \frac{3(x^2 + 5x + 6)}{(x+2)(x-1)} =$$

$$= \lim_{x \to -2} \frac{3\cancel{(x+2)}(x+3)}{\cancel{(x+2)}(x-1)}$$

$$= 3\,\frac{-2+3}{-2-1} = -1$$