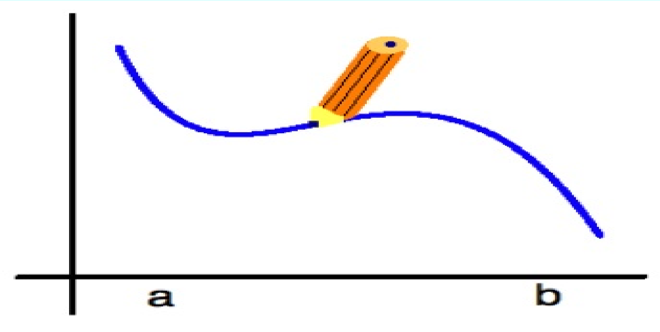
Section 2.5:Continuity

DEFINITION 1. *A function* $f(x)$ *is* **continuous** *at* $x = a$ *if* $\lim_{x\to a} f(x) = f(a)$. *More implicitly: if* f *is continuous at* a *then*

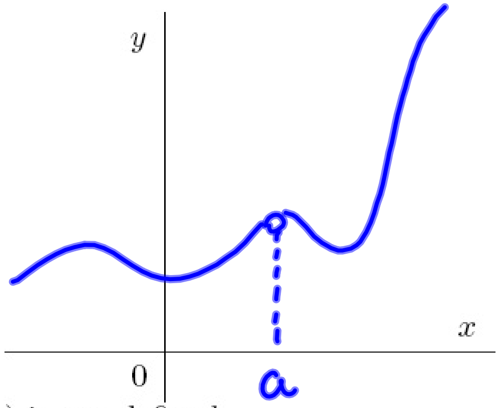
1. $f(a)$ *is defined (i.e.* a *is in the domain of* f*);*
2. $\lim_{x\to a} f(x)$ *exists.*
3. $\lim_{x\to a} f(x) = f(a)$.

A function is said to be continuous on the interval $[a, b]$ *if it is continuous at each point in the interval.*

Geometrically, if f is continuous at any point in an interval then its graph has no break in it (i.e. can be drawn without removing your pen from the paper).

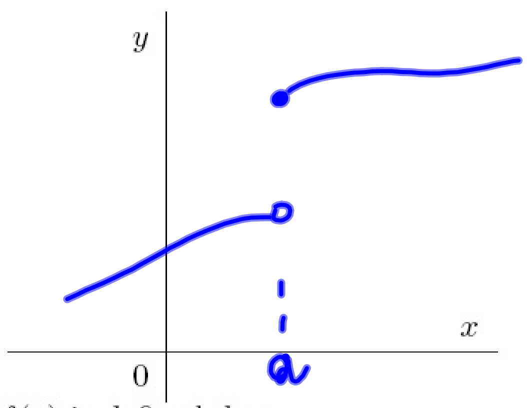


REASONS FOR BEING DISCONTINUOUS:



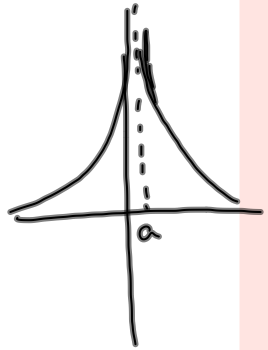
$f(a)$ is not defined

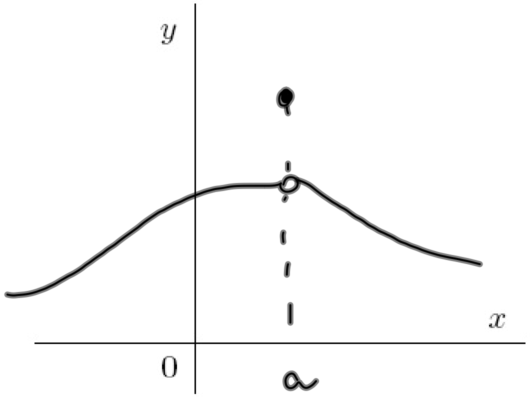
(i.e. a is not in the domain of f)



$f(a)$ is defined, but

the limit as $x \to a$ DNE



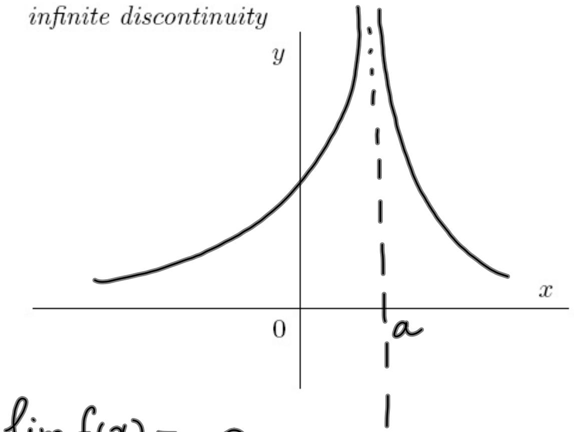


$f(a)$ is defined and $\lim_{x\to a} f(x)$ exists,

but $\lim_{x\to a} f(x) \neq f(a)$

Classification of discontinuities:

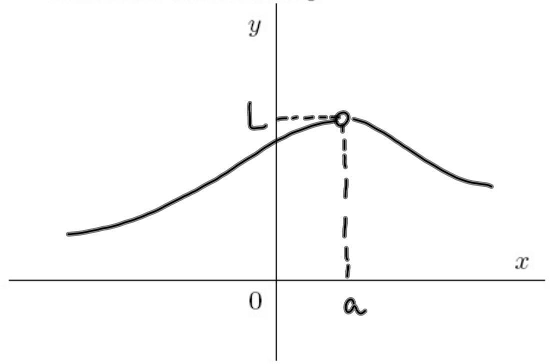
infinite discontinuity



$\lim_{x \to a} f(x) = \infty$

or ONE

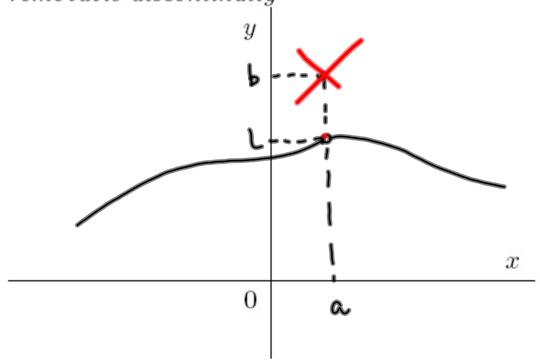
removable discontinuity



$\lim_{x \to a} f(x) = L$

<u>Define</u> $f(a) = L$

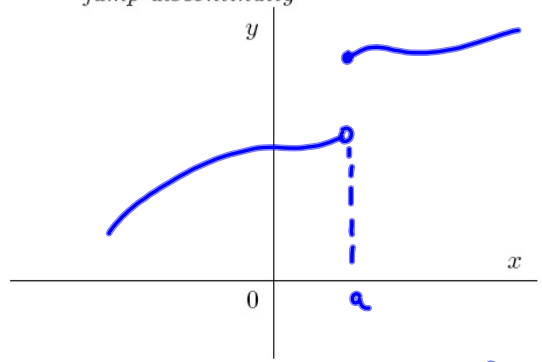
removable discontinuity



Given $\lim_{x \to a} f(x) = L$, but $f(a) = b$

To remove define $f(a) = L$

jump discontinuity



$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

EXAMPLE 2. *Explain why each function is discontinuous at the given point:*

(a) $f(x) = \dfrac{2x}{x-3}$, $\quad x = 3$

f(3) is undefined

(b) $f(x) = \begin{cases} \dfrac{x^2 - 2x + 1}{x-1} & \text{if } x \neq 1 \\ 5 & \text{if } x = 1, \end{cases}$ $\quad x = 1$

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2 - 2x + 1}{x-1} = \lim_{x \to 1} \frac{(x-1)^2}{x-1} = \lim_{x \to 1} x-1 = 0$$

$$0 = \lim_{x \to 1} f(x) \neq f(1) = 5$$

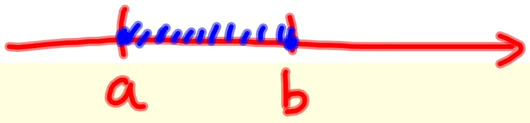
DEFINITION 3. *A function* f *is* **continuous from the right at** $x = a$ *if*

$$\lim_{x \to a^+} f(x) = f(a)$$

and f *is* **continuous from the left at** a *if*

$$\lim_{x \to a^-} f(x) = f(a).$$

REMARK 4. *Functions* ***continuous on an interval*** *if it is continuous at every number in the interval.* At the end point of the interval we understand continuous to mean continuous from the right or continuous from the left.



EXAMPLE 5. *Find the interval(s) where* $f(x) = \sqrt{9 - x^2}$ *is continuous.*

Domain of f : $9 - x^2 \geq 0 \Rightarrow 9 \geq x^2$, $x^2 \leq 9$

$|x| \leq 3$

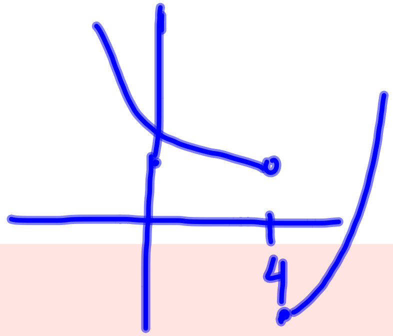
$-3 \leq x \leq 3$

$\Downarrow$

$f(x)$ is cont. on $[-3, 3]$.

EXAMPLE 6. *Find the constant c that makes g continuous on $(-\infty, \infty)$:*

$$g(x) = \begin{cases} x^2 - c^2 & \text{if } \; x < 4 \\ cx^2 - 1 & \text{if } \; x \geq 4 \end{cases}$$



$x^2 - c^2 = cx^2 - 1$ at $x = 4$

$cx^2 - 1 - x^2 + c^2 = 0$

$x^2(c-1) + c^2 - 1 = 0$

$16(c-1) + \underbrace{c^2 - 1} = 0$

$16(c-1) + (c-1)(c+1) = 0$

$(c-1)\left[16 + (c+1)\right] = 0$

$(c-1)(c+17) = 0$

OR

$c^2 + 16c - 17 = 0$

$(c-1)(c+17) = 0$

$\Downarrow$ $\qquad\qquad$ $\Downarrow$

$\boxed{c = 1}$ OR $\boxed{c = -17}$

$(a-b)(a+b)=a^2-b^2$

EXAMPLE 7. *For each of the following, find all discontinuities, classify them by using limits, give the continuity interval(s) for the corresponding function. If the discontinuity is removable, find a function g that agrees with the given function except of the discontinuity point and is continuous at that point.*

(a) $f(x) = \dfrac{x^2-9}{x^4-81}$

Domain of f: $x^4 \neq 81 \Rightarrow x = \pm 3$ discontinuities

Classify discont: $\lim\limits_{x\to\pm3} f(x) = \lim\limits_{x\to\pm3} \dfrac{\cancel{x^2-9}}{\cancel{(x^2-9)}(x^2+9)} = \dfrac{1}{3^2+9} = \dfrac{1}{18}$ exists

$\Rightarrow x = \pm 3$ are removable discont.

Continuity interval $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$

$$g(x) = \begin{cases} f(x), & x \neq \pm 3 \\ \frac{1}{18}, & x = \pm 3 \end{cases} = \frac{1}{x^2+9}$$

(b) $f(x) = \dfrac{7}{x+12}$ Domain $x \neq -12$

$\lim\limits_{x\to-12} f(x)$ DNE, because if $x<-12 \Rightarrow x+12<0 \Rightarrow$

$\Rightarrow \lim\limits_{x\to-12^-} f(x) = -\infty$

Similarly $\lim\limits_{x\to-12^+} f(x) = +\infty$

Conclusion $x=-12$ is infinite discont.

Interval of contin: $(-\infty,-12)\cup(-12,\infty)$

(c) $f(x) = \begin{cases} x^2 + x & \textit{if} \quad x < 2 \\ 8 - x & \textit{if} \quad x > 2 \\ 4 & \textit{if} \quad x = 2 \end{cases}$

Domain is $\mathbb{R}$

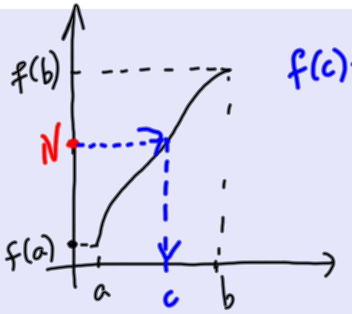
$$\left.\begin{aligned} \lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} x^2 + x = 6 \\ \lim_{x \to 2^+} f(x) &= \lim_{x \to 2^+} 8 - x = 8 - 2 = 6 \end{aligned}\right\} \Rightarrow \lim_{x \to 2} f(x) = 6 \neq f(2) = 4$$

$x = 2$ is remov. disc.

Cont. interval: $(-\infty, 2) \cup (2, +\infty)$

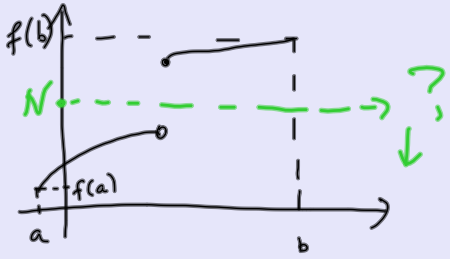
$$g(x) = \begin{cases} x^2 + x, & x < 2 \\ 8 - x, & x > 2 \\ 6. & x = 2 \end{cases}$$

Intermediate Value Theorem: If $f(x)$ is continous on the closed interval $[a, b]$ and N is any number strictly between $f(a)$ and $f(b)$, then there is a number c, $a < c < b$, so that $f(c) = N$.



$f(c)=N$

If $f(x)$ is not cont.



EXAMPLE 8. *If $f(x) = x^5 - 2x^3 + x^2 + 2$, show there a number c so that $f(c) = 1$.*

In other words, Show that the equation $x^5 - 2x^3 + x^2 + 2 = 1$ has a real root.

$f(x)$ is cont. everywhere

To apply Int. Val. Theorem we have to find a & b such that

$f(a) > 1 \qquad f(b) < 1$

x	0	1	-1	-2
$f(x)$	2	2	4	-10
	>1	>1	>1	<1

$\Rightarrow$ By Int. value Theorem on $(-2, -1)$ there is a number c such that $f(c) = 1$.

EXAMPLE 9. *Show that following equation has a solution (a root) between* 1 *and* 2*:*

$$3x^3 - 2x^2 - 2x - 5 = 0.$$

Define $f(x)=3x^3 -2x^2 -2x-5$. This function is continuous for all x.

Look for numbers a and b on [1,2] such that

$f(a)>0$ and $f(b)<0$.

We have $f(1)=-6<0$ and $f(2)=7>0$.

Thus, by Intermidiate Value Theorem there is a number c such that $f(c)=0$, i.e.

the given equation has a solution $x=c$ on the interval [1,2],