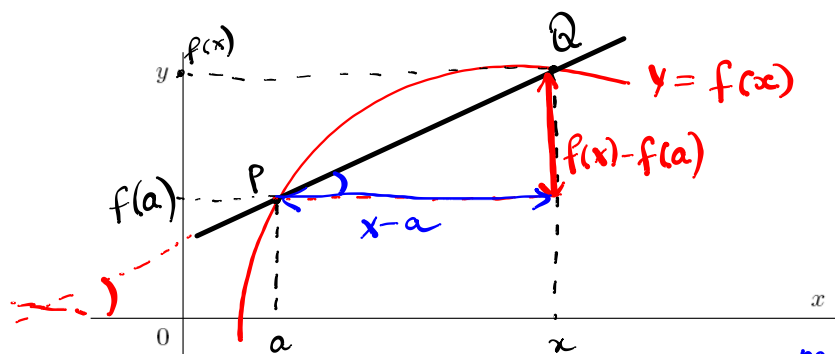


Section 2.7: Tangents, velocities, and other rates of change.

Consider a curve with equation $y = f(x)$ and points $P(a, f(a))$ and $Q(x, f(x))$ on it. The slope of the secant line PQ (also known as average rate or average velocity) is

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$



$$Q \rightarrow P \Rightarrow x \rightarrow a$$

$$m = \lim_{Q \rightarrow P} m_{PQ}$$

DEFINITION 1. The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \quad (1)$$

provided that this limit exists.

The slope (1) of tangent line also known as instantaneous rate of change or instantaneous velocity.

EXAMPLE 2. (a) Find the slope of the tangent line to the graph of $f(x) = x^2 - 3x - 4$ at $(5, 6)$.

$$\begin{aligned} \left. \begin{array}{l} a = 5 \\ f(a) = 6 \end{array} \right\} & m = \lim_{x \rightarrow 5} \frac{f(x) - f(5)}{x - 5} = \lim_{x \rightarrow 5} \frac{x^2 - 3x - 4 - 6}{x - 5} \\ & m = \lim_{x \rightarrow 5} \frac{x^2 - 3x - 10}{x - 5} = \lim_{x \rightarrow 5} \frac{\cancel{(x-5)}(x+2)}{\cancel{x-5}} = 5 + 2 = 7 \end{aligned}$$

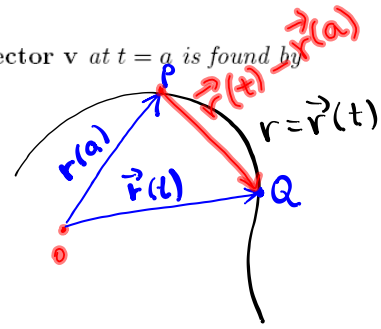
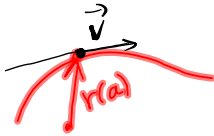
(b) Find the equation of the tangent line to the graph of $f(x)$ at $x = 5$. (Recall that point-slope form for a line through the point (x_1, y_1) with slope m is: $y - y_1 = m(x - x_1)$.)

$$\boxed{y - f(a) = m(x - a)} \quad \begin{array}{l} \text{tangent to } f(x) \\ \text{at } x = a \end{array}$$

$$y - 6 = 7(x - 5)$$

DEFINITION 3. If $\mathbf{r}(t)$ is a vector function then the tangent vector \mathbf{v} at $t = a$ is found by

$$\mathbf{v} = \lim_{t \rightarrow a} \frac{1}{t-a} [\mathbf{r}(t) - \mathbf{r}(a)].$$



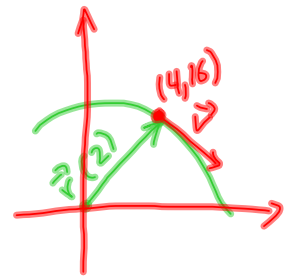
EXAMPLE 4. Given curve $\mathbf{r}(t) = \langle 2t, 10t - t^2 \rangle$.

(a) Find a vector tangent to the curve at the point $(4, 16)$.

Find a : $\mathbf{r}(a) = \langle 2a, 10a - a^2 \rangle = \langle 4, 16 \rangle \Rightarrow \begin{cases} 2a = 4 \Rightarrow a = 2 \\ 10a - a^2 = 16 \end{cases}$ satisfies

$$\mathbf{v} = \lim_{t \rightarrow 2} \frac{1}{t-2} [\mathbf{r}(t) - \mathbf{r}(a)]$$

$$\mathbf{v} = \lim_{t \rightarrow 2} \frac{1}{t-2} [\langle 2t, 10t - t^2 \rangle - \langle 4, 16 \rangle]$$



$$\lim_{t \rightarrow 2} \frac{1}{t-2} \langle 2t - 4, 10t - t^2 - 16 \rangle$$

$$= \lim_{t \rightarrow 2} \langle \cancel{2(t-2)}, -\cancel{(t-2)}(t-8) \rangle = \lim_{t \rightarrow 2} \langle 2, 8-t \rangle = \langle 2, 6 \rangle$$

(b) Find parametric equations of the tangent line to $\mathbf{r}(t)$ at $t = 2$.
 = line through the point $\mathbf{r}(2) = \langle 4, 16 \rangle$ in the direction $\mathbf{v} = \langle 2, 6 \rangle$

$$\begin{aligned} x &= 4 + 2t \\ y &= 16 + 6t \end{aligned}$$

(c) Find a Cartesian equation of the tangent line.
 Way 1 Eliminate parameters

$$\begin{aligned} t &= \frac{x-4}{2} \\ t &= \frac{y-16}{6} \end{aligned} \Rightarrow \boxed{\frac{x-4}{2} = \frac{y-16}{6}}$$

Way 2 Use

$$y - y_1 = m(x - x_1)$$

where $m = \frac{6}{2} = 3$

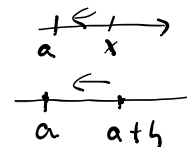
$$\boxed{y - 16 = 3(x - 4)}$$

Velocities. Denote by $f(t)$ the position of an object at time t .
 The **Average Velocity** of the object from $t = a$ to $t = b$ is

$$\frac{f(b) - f(a)}{b - a}.$$

The **Instantaneous Velocity** of the object at time $t = a$ is

$$v(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$



(Compare this formula with (1) by substitution $h = x - a$.)

By (1)

$$v(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

use
 $h = x - a \rightarrow 0$
 \Downarrow
 $x = h + a$

$$v(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

EXAMPLE 5. The position (in meters) of an object moving in a straight path is given by

$$s(t) = t^2 - 2t + 8,$$

where t is measured in seconds.

(a) Find the average velocity over the time interval $[4, 5]$.

$$\text{aver. velocity} = \frac{s(5) - s(4)}{5 - 4} = \frac{25 - 10 + 8 - (16 - 8 + 8)}{1} = 7 \text{ m/s}$$

$16 = s(4)$

(b) Find the instantaneous velocity at time $t = 4$.

$$v(4) = \lim_{t \rightarrow 4} \frac{s(t) - s(4)}{t - 4} = \lim_{t \rightarrow 4} \frac{t^2 - 2t + 8 - 16}{t - 4}$$

$$v(4) = \lim_{t \rightarrow 4} \frac{t^2 - 2t - 8}{t - 4} = \lim_{t \rightarrow 4} \frac{(t - 4)(t + 2)}{t - 4} = 6 \text{ m/s}$$

Other Rates of Change:

The **Average Rate** of change of function $f(x)$ from $x = a$ to $x = b$ is

$$\frac{f(b) - f(a)}{b - a}.$$

The **Instantaneous Rate of Change** of $f(x)$ at $x = a$ is

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$