Section 3.1: Derivative

DEFINITION 1. *The* **Derivative** *of a function* $f(x)$ *at* $x = a$ *is*

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Other common notations for the derivative of $y = f(x)$ *are* f', $\frac{\mathrm{d}}{\mathrm{d}x} f(x)$. $= \frac{df}{dx}$

It follows from the definition that the derivative $f'(a)$ measures:

interpretation of $f'(a)$

- The slope of the tangent line to the graph of $f(x)$ at $(a, f(a))$;
- The instantaneous rate of change of $f(x)$ at $x = a$;
- The instantaneous velocity of the object at time at $t = a$ (if $f(t)$ is the position of an object at time t).

$$f(-3) = \frac{3}{-3+5} = \frac{3}{2}$$

EXAMPLE 2. *Given* $f(x) = \dfrac{3}{x+5}$.

(a) *Find the derivative of* $f(x)$ *at* $x = -3$.

$$f'(-3) = \lim_{x \to -3} \frac{f(x) - f(-3)}{x - (-3)}$$

$$-x-3 = -(x+3)$$

$$f'(-3) = 3\lim_{x \to -3} \frac{\frac{\cancel{3}\,1}{x+5} - \frac{\cancel{3}\,1}{2}}{x+3} = 3 \lim_{x \to -3} \frac{\frac{2-(x+5)}{2(x+5)}}{x+3}$$

$$f'(-3) = 3 \lim_{x \to -3} -\frac{\cancel{x+3}}{2(x+5)\,(\cancel{x+3})} = -3 \lim_{x \to -3} \frac{1}{2(x+5)} = \frac{-3}{2(-3+5)} = -\frac{3}{4}$$

(b) *Find the equation of the tangent line of* $y = f(x)$ *at* $x = -3$.

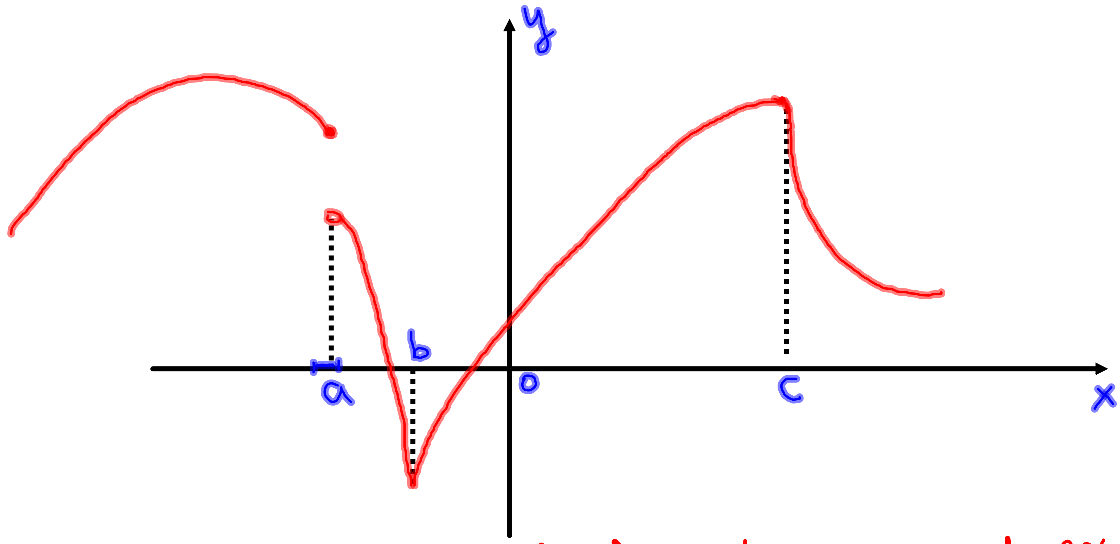
Tangent line to the graph $y = f(x)$ at $x = a$

$$\boxed{y - f(a) = f'(a)(x-a)}$$

$$y - f(-3) = f'(-3)\,(x - (-3))$$

$$\boxed{y - \frac{3}{2} = -\frac{3}{4}(x+3)}$$

Question: Where does a derivative not exist for a function?



Answer: $f'(a)$, $f'(b)$, $f'(c)$ do not exist.

at $x=a$ jump disc. no tangent | we have tangent from left but from right slope $= \infty$

DEFINITION 3. *A function $f(x)$ is said to be* **differentiable** *at $x = a$ if $f'(a)$ exists.*

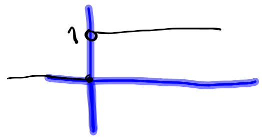
EXAMPLE 4. *Refer to the graph above to determine where f(x) is not differentiable.*

$f(x)$ is not differentiable at $x=a$, $x=b$, and $x=c$.

CONCLUSION: A function $f(x)$ is NOT differentiable at $x = a$ if

- $f(x)$ is not continuous at $x = a$;

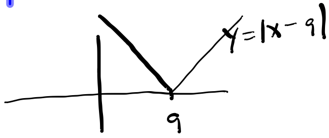
$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x \leq 0 \end{cases}$$



$f(x)$ is not diff. at $x=0$

- $f(x)$ has a sharp turn at $x = a$ (left and right derivatives are not the same);

$$f(x) = |x - 9|$$

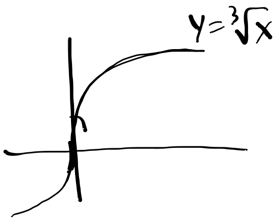


$$f'(9) = \lim_{x \to 9} \frac{f(x) - f(9)}{x - 9} = \lim_{x \to 9} \frac{|x-9| - 0}{x - 9}$$

$$= \lim_{x \to 9} \frac{|x-9|}{x-9} \quad \text{DNE}$$

- $f(x)$ has a vertical tangent at $x = a$.

$$f(x) = \sqrt[3]{x}$$



f is not diff. at $x=0$

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\sqrt[3]{x}}{x} = \lim_{x \to 0} \frac{1}{x^{2/3}} = \infty$$

DNE

tangent is vertical line

THEOREM 5. *If f is differentiable at a then f is continuous at a.*

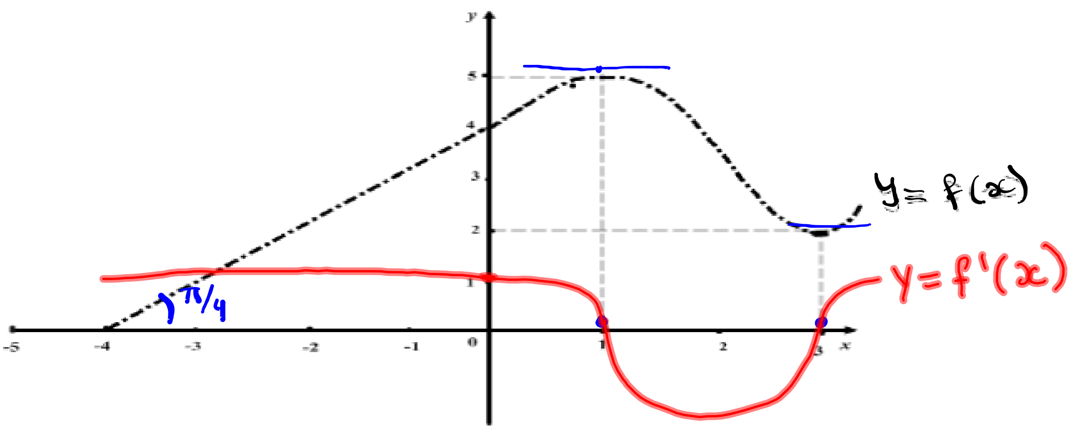
Note If f is continuous $\not\Rightarrow$ f is differ.

The derivative as a function: If we replace a by x in Definition 1 we get:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

A new function $g(x) = f'(x)$ is called the **derivative of f.**

EXAMPLE 6. *Use the graph of $f(x)$ below to sketch the graph of the derivative $f'(x)$.*



EXAMPLE 7. *Use the definition of the derivative to find $f'(x)$ for $f(x) = \sqrt{1+3x}$.*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{1+3(x+h)} - \sqrt{1+3x}}{h} = \lim_{h \to 0} \frac{\sqrt{1+3x+3h} - \sqrt{1+3x}}{h}$$

Multiply by conjugate

$$f'(x) = \lim_{h \to 0} \frac{(\sqrt{\ } - \sqrt{\ })(\sqrt{\ } + \sqrt{\ })}{h(\sqrt{\ } + \sqrt{\ })} = \lim_{h \to 0} \frac{\cancel{1} + \cancel{3x} + 3h - (\cancel{1} + \cancel{3x})}{h(\sqrt{1+3x+3h} + \sqrt{1+3x})}$$

$$f'(x) = \lim_{h \to 0} \frac{3\cancel{h}}{\cancel{h}(\sqrt{1+3x+3h} + \sqrt{1+3x})} = \frac{3}{\sqrt{1+3x+3 \cdot 0} + \sqrt{1+3x}}$$

$$f'(x) = \frac{3}{2\sqrt{1+3x}}, \quad x \neq -\frac{1}{3}$$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

EXAMPLE 8. *Each limit below represents the derivative of function $f(x)$ at $x = a$. State f and a in each case.*

(a) $\lim_{h \to 0} \frac{(3+h)^4 - 81}{h}$ $= \lim_{h \to 0} \frac{(3+h)^4 - 3^4}{h} = f'(3)$

where $f(x) = x^4$

Note that $f(3+h) = (3+h)^4$, $f(3) = 3^4$

(b) $\lim_{x \to \frac{3\pi}{2}} \frac{\sin x + 1}{x - \frac{3\pi}{2}}$ $= \lim_{x \to \frac{3\pi}{2}} \frac{\sin x - \sin \frac{3\pi}{2}}{x - \frac{3\pi}{2}} = f'\left(\frac{3\pi}{2}\right)$

$-(-1)$

where $f(x) = \sin x$

Note that $\sin \frac{3\pi}{2} = -1$