## Section 3.10: Related rates

In this section, we have two or more quantities that are changing with respect to time $t$. We will apply the following strategy:

1. Read the problem carefully and draw a diagram if possible. (units)
2. Express the given information and the required rates in terms of derivatives and state your "find" and "when".
3. Find a formula (equation) that relates the quantities in the problem. (If necessary, use Geometry[1] of the situation to eliminate one of the variables by substitution.) **Don't substitute the given numerical information at this step!!!**
4. Use the Chain Rule to differentiate both sides of the equation with respect to $t$.
5. Substitute the given numerical information in the resulting equation and solve for the desired rate of change.

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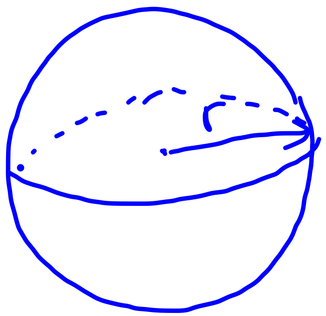
[1] Useful formulas:

- Triangle: $A = \frac{1}{2}bh$
  - Equilateral Triangle: $h = \dfrac{\sqrt{3}}{2}s$; $A = \dfrac{\sqrt{3}s^2}{4}$
  - Right Triangle: Pythagorean Theorem $c^2 = a^2 + b^2$
- Trapezoid: $A = \dfrac{h}{2}(b_1 + b_2)$
- Parallelogram: $A = bh$
- Circle: $A = \pi r^2$; $C = 2\pi r$
- Sector of Circle: $A = \frac{1}{2}r^2\theta$; $s = r\theta$
- Sphere: $V = \dfrac{4}{3}\pi r^3$; $A = 4\pi r^2$
- Cylinder: $V = \pi r^2 h$
- Cone: $V = \dfrac{1}{3}\pi r^2 h$

EXAMPLE 1. *A spherical balloon is inflated with gas at a rate of* $25\text{ft}^3/\text{min}$. *How fast is the radius changing when the radius is* $2\text{ft}$?

Quantities that are changing: $V$, $r$




Given $\frac{dV}{dt} = 25 \text{ ft}^3/\text{min}$

Find $\left.\frac{dr}{dt}\right|_{r=2\text{ft}}$ - ?

$V = \frac{4}{3}\pi r^3$ (don't substitute $r=2$)

$V(t) = \frac{4}{3}\pi [r(t)]^3$

$\frac{d}{dt}[V(t)] = \frac{4}{3}\pi \frac{d}{dt}[r(t)]^3$

$\frac{dV}{dt} = \frac{4}{\cancel{3}}\pi \cancel{3} r^2 \frac{dr}{dt}$

$\left.\frac{dr}{dt}\right|_{r=2} = \left.\frac{1}{4\pi r^2}\frac{dV}{dt}\right|_{r=2} = \frac{1}{4\pi\cdot 2^2}\cdot 25 = \boxed{\frac{25}{16\pi}\frac{\text{ft}}{\text{min}}}$

**Problem** If a snowball melts so that its surface area decreases at a rate 1cm$^2$/min, find the rate at which the diameter decreases when the diameter is 10cm.

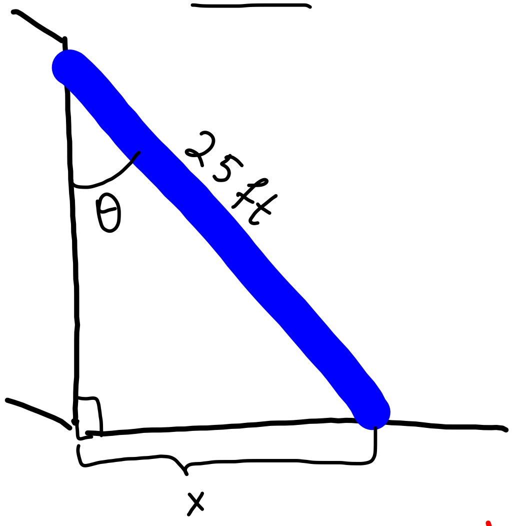
Solution: The quantities which are changing they are surface area, A, and diameter, D.

Given: $A'(t)=-1$cm$^2$/min. Find $D'(t)$ when $D=10$cm.

We know that $D=2R$(where R is radius). Thus $A=4\pi R^2=4\pi(D/2)^2=\pi D^2$,
or $A(t)=\pi[D(t)]^2$. Differentiating the last formula we get:
$A'(t)=2\pi D(t)D'(t)$. Plug in the given numerical data and get
$-1=2\pi 10D'(t)$, or $D'(t)=-1/(20\pi)$ cm/min.

Answer: The rate of decrease is $1/(20\pi)$ cm/min.

EXAMPLE 2. *A ladder* 25 *feet long and leaning against a vertical wall. The bottom of the ladder slides away from the wall at speed* 3 feet/sec. *Determine how fast the angle between the top of the ladder and the wall is changing when the angle is* $\frac{\pi}{4}$ *radians.*




Quantities $x$, $\theta$

$$\frac{dx}{dt} = 3$$

Find $\left.\frac{d\theta}{dt}\right|_{\theta=\frac{\pi}{4}} = ?$

$$\frac{d}{dt}\left(\sin\theta(t)\right) = \frac{d}{dt}\left(\frac{x(t)}{25}\right)$$

$$\cos\theta \, \frac{d\theta}{dt} = \frac{1}{25}\frac{dx}{dt}$$

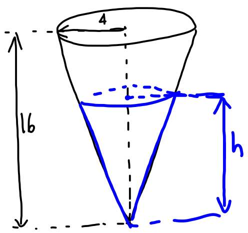
$\theta = \frac{\pi}{4} \Rightarrow$ $$\underbrace{\cos\frac{\pi}{4}}_{\frac{1}{\sqrt{2}}} \frac{d\theta}{dt} = \frac{1}{25}\cdot 3$$

$$\frac{d\theta}{dt} = \boxed{\frac{3\sqrt{2}}{25} \text{ rad/sec}}$$

EXAMPLE 3. *A water tank has the shape of an inverted right circular cone with height 16m and base radius 4m. Water is pouring into the tank at* $3\text{m}^3/\text{min}$.

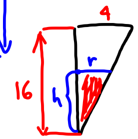
(a) *How fast is the water level rising when the water in the tank is 5 meters deep?*

Quantities: $V$, $h$




$\frac{dV}{dt} = 3$

$\left.\frac{dh}{dt}\right|_{h=5}$




$V = \frac{1}{3}\pi r^2 h$

$\frac{16}{h} = \frac{4}{r}$

$16r = 4h$

$4r = h \Rightarrow \boxed{r = \frac{h}{4}}$

use in (b)

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{4}\right)^2 h = \frac{\pi}{48} h^3$$

$$\frac{d}{dt}\left(V(t)\right) = \frac{\pi}{48}\frac{d}{dt}\left((h(t))^3\right)$$

$$\frac{dV}{dt} = \frac{\pi}{48} 3h^2 \frac{dh}{dt}$$

$$h=5 \Rightarrow \cancel{3} = \frac{\pi}{48}\cdot\cancel{3}\cdot 25\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \boxed{\frac{48}{25\pi}\ \text{m/min}}$$

(b) *At what rate is the radius of the top of the water in the tank changing when the depth of the water is 5 meters?*

Quantities: $V$, $r$

$\frac{dV}{dt} = 3$ Find $\left.\frac{dr}{dt}\right|_{h=5}$

$$V = \frac{1}{3}\pi r^2 h \Rightarrow V = \frac{1}{3}\pi r^2 4r$$

$h = 4r$ (see above)

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{4}{\cancel{3}}\pi\, \cancel{3}r^2\frac{dr}{dt}$$

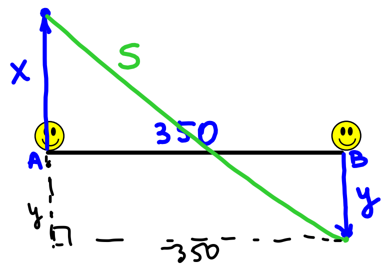
when $h=5$ we have $r = \frac{h}{4} = \frac{5}{4}$

$$3 = 4\pi r^2\frac{dr}{dt}$$

$$\left.\frac{dr}{dt} = \frac{3}{4\pi r^2}\right|_{h=5}$$

$$\frac{dr}{dt} = \frac{3}{\cancel{4}\pi\cdot\frac{5^2}{4^{\cancel{2}}}} = \frac{12}{25\pi}\ \text{m/min}$$

EXAMPLE 4. *Two people are separated by 350 meters. Person A starts walking north at a rate of 0.6 m/sec and 7 minutes later Person B starts walking south at 0.5 m/sec. At what rate is the distance separating the two people changing 25 minutes after Person A starts walking?*




$25\text{min} = 25\cdot 60\,s = 1500\,s$

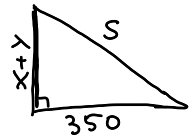
$\frac{dx}{dt} = 0.6\ m/s$

$\frac{dy}{dt} = 0.5\ m/s$

$\left.\frac{ds}{dt}\right|_{t=1500}$

Quantities

x, y, s




$s^2 = 350^2 + (x+y)^2$

$$\frac{d}{dt}\left[(s(t))^2\right] = \frac{d}{dt}\left[350^2 + [x(t)+y(t)]^2\right]$$

$$\left.\cancel{2}\, s(t)\frac{ds}{dt}\right|_{t=1500} = \left.\cancel{2}\,(x(t)+y(t))\left(\frac{dx}{dt}+\frac{dy}{dt}\right)\right|_{t=1500}$$

$$x(1500) = (\text{speed}_A)(\text{time A}) = \frac{dx}{dt}\cdot 1500 = 0.6\cdot 1500 = 900\ m$$

$$y(1500) = (\text{speed B})(\text{time B}) = 0.5\cdot(25-7)\cdot 60 = 540\ m$$

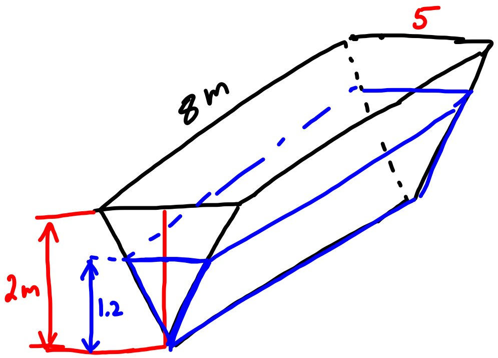
$$s(1500) = \left.\sqrt{350^2+(x+y)^2}\right|_{t=1500} = \sqrt{350^2+(900+540)^2} \approx 1482\ m$$

$$1482\frac{ds}{dt} \approx (900+540)\cdot(0.6+0.5)$$

$$\boxed{\frac{ds}{dt} \approx 1.07\ m/s}$$

EXAMPLE 5. *A trough of water is 8 meters long and its ends are in the shape of isosceles triangles whose width is 5 meters and height is 2 meters. If the trough is filled with water at a constant rate of* $6\text{m}^3/\text{s}$, *how fast the water level (the height of the water) changing when the water is 120cm deep?*

1.2 m



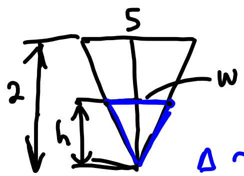

Quantities: $V, h$

Given $\frac{dV}{dt} = 6 \; m^3/s$

Find $\left.\frac{dh}{dt}\right|_{h=1.2\text{ m}} = ?$

$V = (\text{Area of end}) \cdot \text{length}$

$\text{Area of end} = \frac{\text{Base} \cdot \text{height}}{2}$, $\text{length} = 8$




$\Delta \sim \Delta$

$V = \frac{wh}{2} \cdot 8 = 4wh$

$\frac{5}{w} = \frac{2}{h} \Rightarrow 5h = 2w \Rightarrow w = \frac{5h}{2}$

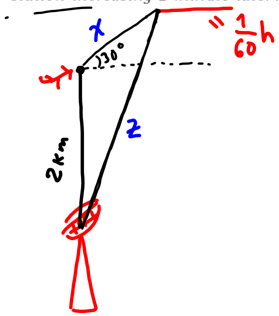
$\Rightarrow V = 4 \, \frac{5h}{2} \cdot h = 10h^2$

$\frac{d}{dt} V(t) = \frac{d}{dt} 10 [h(t)]^2$

$\frac{dV}{dt} = 10 \cdot 2h \frac{dh}{dt}$

$h = 1.2 \Rightarrow 6 = 20 \cdot 1.2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = 0.25 \; m/s$

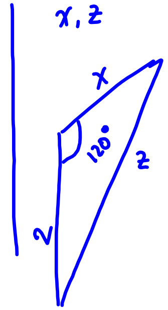
EXAMPLE 6. *A plane flying with a constant speed of 360km/hour passes over a radar station at an altitude of 2km and climbs at an angle of 30°. At what rate is the distance from the plane to the radar station increasing 1 minute later?*




$$\frac{dx}{dt} = 360 \text{ km/h}$$

$$\left.\frac{dz}{dt}\right|_{t=\frac{1}{60}h} = ?$$

x, z




Use the Law of Cosines

$$z^2 = x^2 + 2^2 - 2 \cdot 2 \cdot x \cos 120°$$

$$z^2 = x^2 + 4 - 4x\left(-\frac{1}{2}\right)$$

$$z^2 = x^2 + 4 + 2x$$

Differentiate

$$\frac{d}{dt}[z(t)]^2 = \frac{d}{dt}\left[[x(t)]^2 + 4 + 2x(t)\right]$$

$$2z(t)\frac{dz}{dt} = 2x(t)\frac{dx}{dt} + 0 + 2\frac{dx}{dt}$$

$t = \frac{1}{60}$

$$x\left(\frac{1}{60}\right) = (\text{speed})(\text{time}) = 360 \cdot \frac{1}{60} = 6 \text{ km}$$

$$z\left(\frac{1}{60}\right) = \sqrt{\left[x\left(\tfrac{1}{60}\right)\right]^2 + 4 + 2x\left(\tfrac{1}{60}\right)}$$

$$= \sqrt{6^2 + 4 + 2 \cdot 6} = 2\sqrt{13} \text{ km}$$

$$\cancel{2} \cdot \cancel{2}\sqrt{13}\frac{dz}{dt} = \cancel{2} \cdot 6 \cdot \overset{180}{\cancel{360}} + \cancel{2} \cdot \overset{180}{\cancel{360}}$$

$$\frac{dz}{dt} = \frac{7 \cdot 180}{\sqrt{13}} \approx 350 \text{ km/h}$$

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