

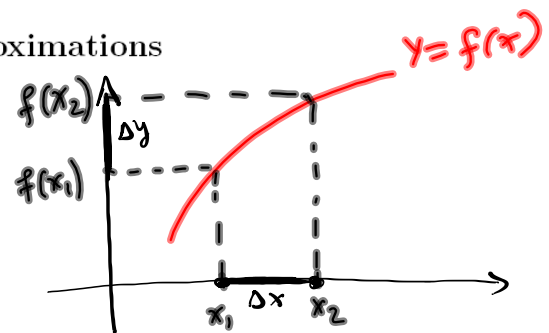
3.11: Differentials; Linear and Quadratic Approximations

Differentials: If x changes from x_1 to x_2 , then the *change in x* is

$$\Delta x = x_2 - x_1.$$

If $y = f(x)$ then corresponding *change in y* is

$$\Delta y = f(x_2) - f(x_1).$$



DEFINITION 1. Let $y = f(x)$, where f is a differentiable function. Then the differential dx is an independent variable (i.e. dx can be given the value of any real number). The differential dy is then defined in terms of dx by the equation

$$dy = f'(x)dx = y'dx$$

EXAMPLE 2. Compare the values of Δy and dy if

$$y = f(x) = x^2 - 2x$$

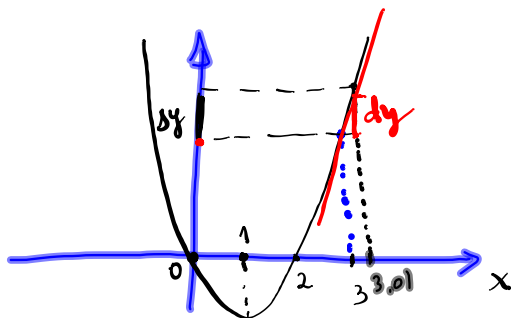
and x changes from 3 to 3.01. Illustrate these quantities graphically.

$$\Delta x = x_2 - x_1 = 3.01 - 3 = 0.01$$

$$dx \approx \Delta x = 0.01$$

$$\begin{aligned} \Delta y &= y(x_2) - y(x_1) = f(3.01) - f(3) = \\ &= \underline{3.01^2} - 2 \cdot 3.01 - (\underline{3^2} - 2 \cdot 3) = \boxed{0.0401} \end{aligned}$$

$$dy(3) = y'(3) dx = (2x - 2) \Big|_{x=3} \cdot 0.01 = (6 - 2) \cdot 0.01 = \boxed{0.04}$$



$$y = f(x) = x^2 - 2x$$

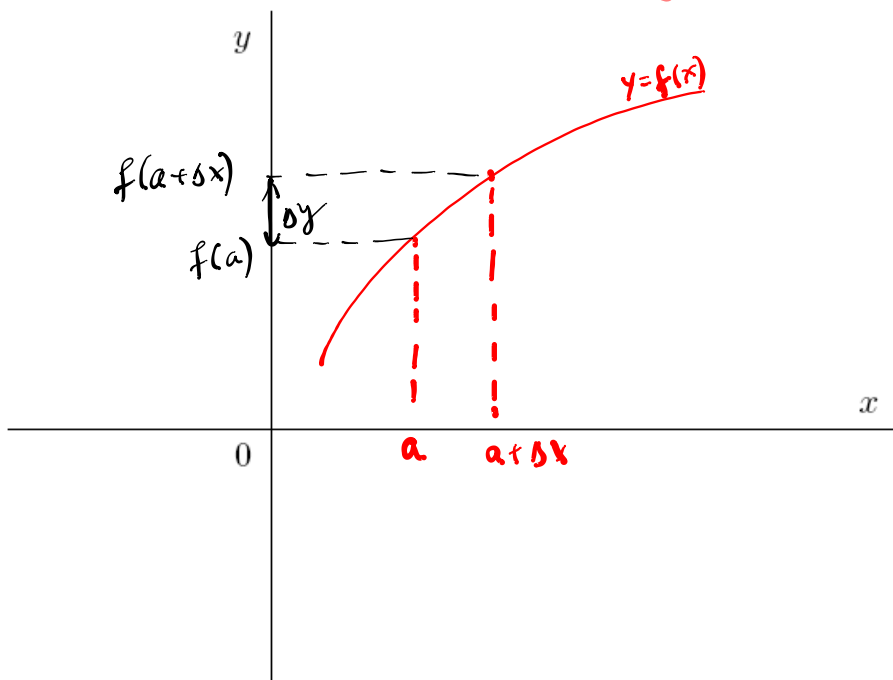
REMARK 3. Notice that dy was easier to compute than Δy . For more complicated functions (for example $y = \cos x$) it may be impossible to compute Δy exactly.

CONCLUSION: $\Delta y \approx dy$ provided we keep Δx small. This yields the following “approximation by differentials” formula:

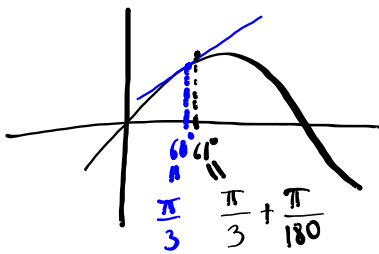
$$f(a + \Delta x) \approx f(a) + dy = f(a) + f'(a)\Delta x.$$

Indeed, assume x changes from $x = a$ to $x = a + \Delta x$. Then

$$f'(a) dx = dy \approx \Delta y = f(a + \Delta x) - f(a)$$



EXAMPLE 4. Use differentials to find an approximate value for $\sin 61^\circ$.



$$f(x) = \sin x, \quad a = \frac{\pi}{3}, \quad \Delta x = \frac{\pi}{180}$$

$$f(a + \Delta x) \approx f(a) + f'(a) \Delta x$$

$$f(a) = f\left(\frac{\pi}{3}\right) = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$f'(x) = \cos x$$

$$f'(a) = \cos \frac{\pi}{3} = \frac{1}{2}$$

$$\sin 61^\circ = f\left(\frac{\pi}{3} + \frac{\pi}{180}\right) \approx \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{\pi}{180} \approx 0.87461$$

Note using calculator $\sin 61^\circ \approx 0.87475$

EXAMPLE 5. Use differentials to find an approximate value for $\sqrt[4]{0.98}$.

$$f(x) = \sqrt[4]{x}, \quad a = 1, \quad \Delta x = -0.02$$

$$f(a + \Delta x) \approx f(a) + f'(a) \Delta x$$

$$f(a) = f(1) = \sqrt[4]{1} = 1$$

$$f'(x) = (\sqrt[4]{x})' = (x^{\frac{1}{4}})' = \frac{1}{4} x^{-\frac{3}{4}}$$

$$f'(a) = \frac{1}{4} (1)^{-\frac{3}{4}} = \frac{1}{4}$$

$$\sqrt[4]{0.98} = f(1 - 0.02) \approx 1 + \frac{1}{4} \cdot (-0.02) = 0.995$$

Note: using calculator $\sqrt[4]{0.98} \approx 0.99496$

EXAMPLE 6. A sphere was measured and its radius was found to be 15 inches with a possible error of no more than 0.02 inches. What is the maximum possible error and what is the relative error in the volume if we use this value of the radius?

$$r = 15, \quad \Delta r = 0.02$$

$$r = 15 \pm 0.02$$

$$\Delta V \Big|_{r=15} = ?$$

$$\frac{\Delta V}{V} \Big|_{r=15} = ?$$

$$V(r) = \frac{4}{3} \pi r^3 \Rightarrow V'(r) = \frac{4}{3} \pi \cdot 3r^2 = 4\pi r^2$$

$$\Delta V \approx dV = V'(r) dr = 4\pi r^2 dr = 4\pi r^2 \Delta r$$

$$\Delta V(15) = 4\pi \cdot 15^2 \cdot 0.02 = \boxed{56.57 \text{ in}^3}$$

$$4\pi \cdot 225 \cdot 0.02 = 4\pi \cdot 4.5 = 18\pi \approx 18 \cdot 3.14$$

$$V(15) = \frac{4}{3} \pi \cdot 15^3 = 14137.17 \text{ in}^3$$

Note: actual volume when $r=15$
 $V = 14137.17 \pm 56.57 \text{ in}^3$

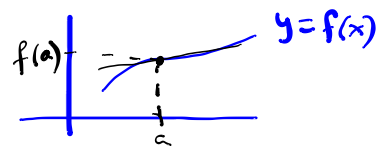
Relative error:

$$\frac{\Delta V(15)}{V(15)} = \frac{4\pi r^2 \Delta r}{\frac{4}{3}\pi r^3} \Big|_{r=15} = \frac{3 \Delta r}{r} \Big|_{r=15} = \frac{3 \cdot 0.02}{15} = 0.004$$

Conclusion: the relative error in radius: $\Delta r/r = 0.013 = 1.3\%$
 produces a relative error in volume $0.004 = 0.4\%$.

Linear Approximation: The function

$$y = L_a(x) = L(x) = f(a) + f'(a)(x - a)$$



(whose graph is the tangent line to the curve $y = f(x)$ at $(a, f(a))$) is called the linearization of f at

a . The approximation

$$f(x) \approx L(x)$$

$$f(x) - f(a) = f'(a)(x - a)$$

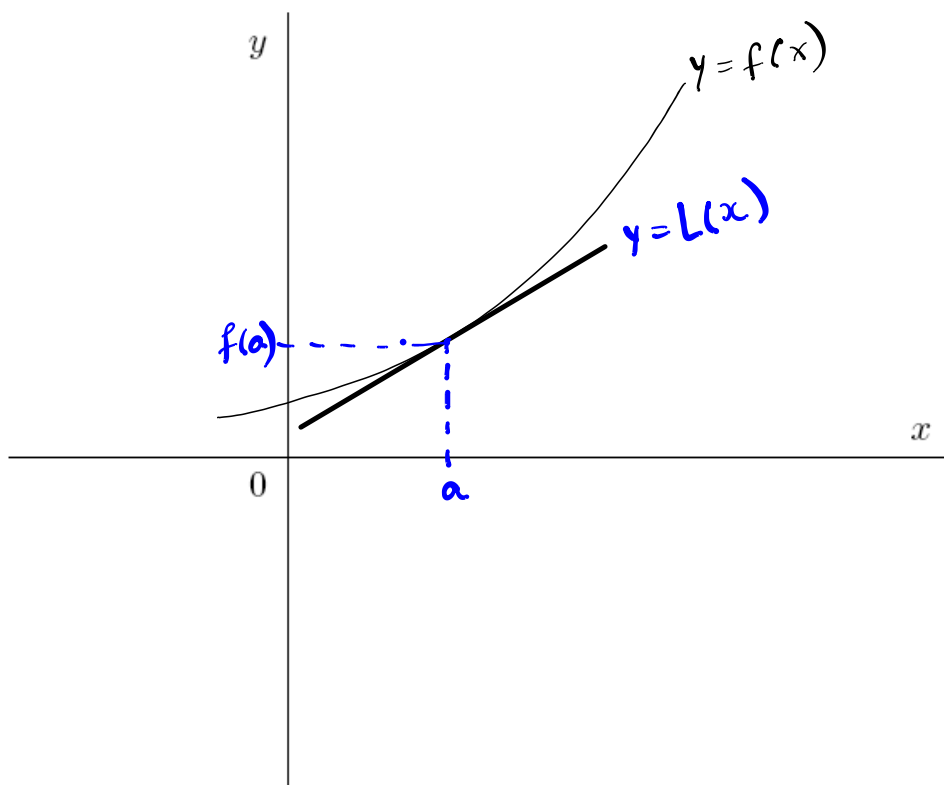
$$f(x) \approx f(a) + f'(a)(x - a)$$

is called the linear approximation or tangent line approximation of f at a .

$$x = a + \Delta x$$

$$x - a = \Delta x$$

$$f(a + \Delta x) \approx f(a) + f'(a)\Delta x$$



EXAMPLE 7. Determine the linearization for $f(x) = \sqrt[3]{x+1}$ at $a = 7$. Use the linear approximation to approximate the values of $\sqrt[3]{8.05}$.

$$f(x) = \sqrt[3]{x+1}, \quad a = 7$$

$$\sqrt[3]{8.05} = \sqrt[3]{7.05+1} = f(7.05)$$

$$L(x) = f(a) + f'(a)(x-a)$$

$$f(a) = f(7) = \sqrt[3]{7+1} = \sqrt[3]{8} = 2$$

$$f'(x) = (\sqrt[3]{x+1})' = ((x+1)^{\frac{1}{3}})' = \frac{1}{3} (x+1)^{-\frac{2}{3}}$$

$$f'(a) = f'(7) = \frac{1}{3} (1+7)^{-\frac{2}{3}} = \frac{1}{3} \left(8^{\frac{1}{3}}\right)^{-2} = \frac{1}{3} 2^{-2} = \frac{1}{12}$$

$$L(x) = 2 + \frac{1}{12}(x-7)$$

Note that $y = L(x)$ is tangent line to the graph of $f(x)$ at $a=7$

$$\sqrt[3]{8.05}$$

$$\approx f(7.05) \approx L(7.05) = 2 + \frac{1}{12}(7.05-7)$$

$$= 2 + \frac{1}{12} \cdot 0.05 \approx 2.004167$$

Note: using calculator $\sqrt[3]{8.05} \approx 2.004158$

EXAMPLE 8. Determine the linear approximation for $\sin x$ at $a = 0$.

$$f(x) = \sin x$$

$$f'(x) = \cos x$$

$$L(x) = f(0) + f'(0)(x - 0) = 0 + 1 \cdot x = \boxed{x}$$

Note

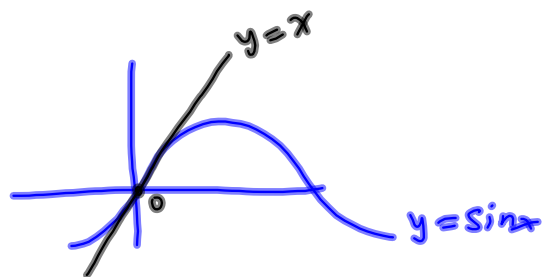
$$\boxed{\sin x \approx x}$$

Note



$$\frac{\sin x}{x} \approx 1 \text{ near } x=0$$

$$\text{cf. } \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$



$$Q(x) = L(x) + \frac{f''(a)}{2}(x-a)^2$$

Quadratic Approximation: If f is a twice differentiable function, then the quadratic approximation to $f(x)$ near a is

$$f(x) \approx \overbrace{f(a) + f'(a)(x-a)}^{L(x)} + \frac{f''(a)}{2}(x-a)^2 = Ax^2 + Bx + C = y$$

This formula can be obtained approximating a curve $y = f(x)$ by a parabola instead by a straight line.

EXAMPLE 9. Find the quadratic approximation to $\cos x$ near $a = 0$.

$$\begin{array}{l|l} f(x) = \cos x & f(0) = 1 \\ f'(x) = -\sin x & f'(0) = 0 \\ f''(x) = -\cos x & f''(0) = -1 \end{array}$$

$$Q(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2$$

$$Q(x) = 1 + 0 \cdot x - \frac{1}{2}x^2$$

$$Q(x) = 1 - \frac{1}{2}x^2$$

$y = 1 - \frac{1}{2}x^2$ parabola

Linearization: $L(x) = 1$

$$y = 1$$

