Section 3.2: Differentiation formulas

The properties and formulas in this section will be given in both "prime" notation and "fraction" notation.

PROPERTIES:

1. *Constant rule*: If f is a constant function, $f(x) = c$, then $f'(x) = 0$, or $\frac{\mathrm{d}c}{\mathrm{d}x} = 0$.

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{c-c}{h} = 0$ $\quad (5)' = 0$

2. *Power rule:* If $f(x) = x^n$, where n is a real number, then $f'(x) = nx^{n-1}$, or $\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$.

$\frac{d}{dx}(x^{-2012}) = -2012\, x^{-2012-1} = 2012\, x^{-2013}$

$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = -1 \cdot x^{-1-1} = -\frac{1}{x^2}$ $\quad \boxed{\left(\frac{1}{x}\right)' = -\frac{1}{x^2}}$

$\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2\sqrt{x}}$ $\quad \boxed{(\sqrt{x})' = \frac{1}{2\sqrt{x}}}$

$= \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{1/2}} =$

3. *Constant multiple rule:* If c is a constant and $f'(x)$ exists then

$$(cf(x))' = cf'(x), \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}x}(cf) = c\frac{\mathrm{d}f}{\mathrm{d}x}.$$

$(3\sin x)' = 3(\sin x)'$

4. *Sum/Difference rule*: If $f'(x)$ and $g'(x)$ exists then

$$(f(x) + g(x))' = f'(x) + g'(x), \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}x}(f+g) = \frac{\mathrm{d}f}{\mathrm{d}x} + \frac{\mathrm{d}g}{\mathrm{d}x}.$$

5. *Product rule:* If $f'(x)$ and $g'(x)$ exists then

$$\boxed{(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)}, \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}x}(fg) = f\frac{\mathrm{d}g}{\mathrm{d}x} + g\frac{\mathrm{d}ff}{\mathrm{d}x}.$$

$(fg)' = f'g + fg'$ $\quad$ Note $(fg)' \neq f' \cdot g'$

6. *Quotient rule:* If $f'(x)$ and $g'(x)$ exists then $, g'(x) \neq 0$

$$\boxed{\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}} \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{\mathrm{d}}{\mathrm{d}x}f(x) - f(x)\frac{\mathrm{d}}{\mathrm{d}x}g(x)}{[g(x)]^2}.$$

$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

EXAMPLE 1. *Find the derivatives of the following functions:*

(a) $f(x) = x^{10} + 3x^5 - 12x + 44 - \pi^5$

$f'(x) = (x^{10})' + 3(x^5)' - 12\, x' + (44 - \pi^5)'$

$f'(x) = 10x^9 + 3 \cdot 5x^4 - 12 + 0 = 10x^9 + 15x^4 - 12$

(b) $g'(t) = \left((1 + \sqrt{t})^2\right)' = \left(1 + 2\sqrt{t} + t\right)' = 0 + 2\,\frac{1}{2\sqrt{t}} + 1 = \frac{1}{\sqrt{t}} + 1$

(c) $F(s) = \left(\frac{s}{3}\right)^4 - s^{-5} = \frac{1}{81}\, s^4 - s^{-5}$

$F'(s) = \frac{1}{81} \cdot 4s^3 - (-5)s^{-5-1} = \frac{4}{81}s^3 + 5s^{-6}$

(d) $y = \frac{u^5+1}{u^2\sqrt{u}} = \frac{u^5+1}{u^{5/2}} = u^{\frac{5}{2}} + u^{-\frac{5}{2}}$

$u^2\sqrt{u} = u^2 u^{\frac{1}{2}} = u^{\frac{2+1}{2}}$
$= u^{5/2}$

$y' = \frac{5}{2} u^{\frac{5}{2}-1} + \left(-\frac{5}{2}\right) u^{-\frac{5}{2}-1} = \frac{5}{2}\left[u^{\frac{3}{2}} - u^{-\frac{7}{2}} \right]$

$\frac{u^5}{u^{5/2}} + \frac{u^0}{u^{5/2}}$
$u^{5-5/2} + u^{0-5/2}$

(e) $f(x) = \overbrace{(x^4 - 3x^2 + 11)}^{u}\overbrace{(3x^3 - 5x^2 + 22)}^{v} = uv$

$(uv)' = u'v + uv'$

$f'(x) = (4x^3 - 6x)(3x^3 - 5x^2 + 22) + (x^4 - 3x^2 + 11)\cdot(9x^2 - 10x)$

(f) $g(z) = \frac{4-z^2}{4+z^2} = \frac{u}{v} \Rightarrow g' = \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

$g'(z) = \frac{(4-z^2)'(4+z^2) - (4-z^2)(4+z^2)'}{(4+z^2)^2}$

$g'(z) = \frac{-2z(4+z^2) - (4-z^2)\cdot 2z}{(4+z^2)^2}$

EXAMPLE 2. *The functions f and g satisfy the properties as shown in the table below:*

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	-5	8	5	12
3	1	2	-2	8

Find the indicated quantity:

(a) $h'(3)$ *if* $h(x) = (3x^2+1)g(x)$ Use Product Rule

$$h'(x) = (3x^2+1)'g(x) + (3x^2+1)g'(x)$$

$$h'(x) = 6x\, g(x) + (3x^2+1)g'(x)$$

$$h'(3) = 18\, g(3) + 28\, g'(3) = 18\cdot(-2) + 28\cdot 8 =$$

$$= -36 + 224 = 188$$

(b) $H'(1)$ *if* $H(x) = \dfrac{x^2}{f(x)}$ Quotient Rule

$$H'(x) = \frac{(x^2)'f(x) - x^2 f'(x)}{[f(x)]^2} = \frac{2x\, f(x) - x^2 f'(x)}{[f(x)]^2}$$

$$H'(1) = \frac{2f(1) - f'(1)}{[f(1)]^2} = \frac{2\cdot(-5) - 8}{(-5)^2} = -\frac{18}{25}$$

EXAMPLE 3. *Given* $f(x) = x^3 - 5x^2 + 6x - 3$

(a) *Find the equation of the tangent line to the graph of* $f(x)$ *at the point* $(1, -1)$.

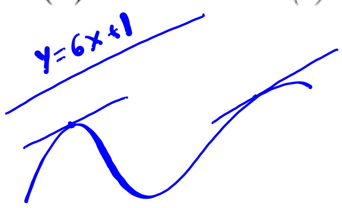
$$y - f(a) = f'(a)(x - a)$$ Tangent to $y = f(x)$ at $x = a$

$a = 1$, $f(1) = -1$; $f'(x) = 3x^2 - 10x + 6$

$f'(1) = 3 - 10 + 6 = -1$

$$\boxed{y + 1 = -(x - 1)}$$

(b) *Find the value(s) of* x *where* $f(x)$ *has a tangent line that is parallel to* $y = 6x + 1$.



Denote the tangent point by $(a, f(a))$

$f'(a) = 6$

$\cancel{y - f(a) = 6 \cdot (x - a)}$

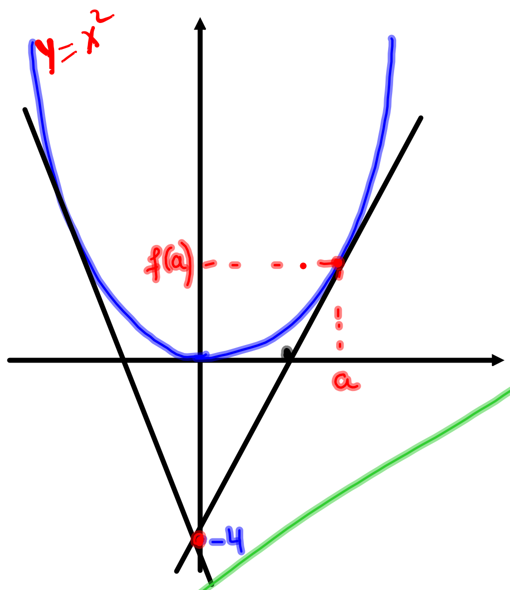
$f'(a) = 3a^2 - 10a + 6 = 6$

$3a^2 - 10a = 0$

$a(3a - 10) = 0$

$\boxed{a = 0}$ or $\boxed{a = \frac{10}{3}}$

EXAMPLE 4. *Show that there are two tangent lines to the parabola $y = x^2$ that pass through the point $(0, -4)$ and find their equations.*



$y - f(a) = f'(a)(x-a)$

Find a (according to picture there are 2 such values).

$f(a) = a^2$

$f'(a) = 2a$

$y - a^2 = 2a(x-a)$

This line contains $(0,-4)$ $\Rightarrow$

$-4 - a^2 = 2a(0-a)$

$-4 - a^2 = -2a^2$

$-4 = -a^2$

$a^2 = 4 \Rightarrow \boxed{a = \pm 2}$

Tangent lines equations:

$$\boxed{\begin{aligned} y - 4 &= 4(x-2) \\ y - 4 &= -4(x+2) \end{aligned}}$$

for $a=2$

for $a=-2$

EXAMPLE 5. *Let* $f(x) = \begin{cases} -1-2x & \text{if} \quad x < -1 \\ x^2 & \text{if} \quad -1 \le x \le 1 \\ x & \text{if} \quad x > 1 \end{cases}$

polynomials

(a) *Give a formula for* f'.

$$f'(x) = \begin{cases} -2 \, , & x < -1 \\ 2x \, , & -1 < x < 1 \\ 1 \, , & x > 1 \\ -2 \, , & x = -1 \\ \text{DNE} \, , & x = 1 \end{cases} \Rightarrow \boxed{f'(x) = \begin{cases} -2 \, , & x \le -1 \\ 2x \, , & -1 \le x < 1 \\ 1 \, , & x > 1 \end{cases}}$$

$x < -1 \quad f'_-(-1) = -2$

$\qquad\qquad\qquad\qquad \parallel \quad \Rightarrow f'(-1) = -2$

$x > -1 \quad f'_+(-1) = -2$

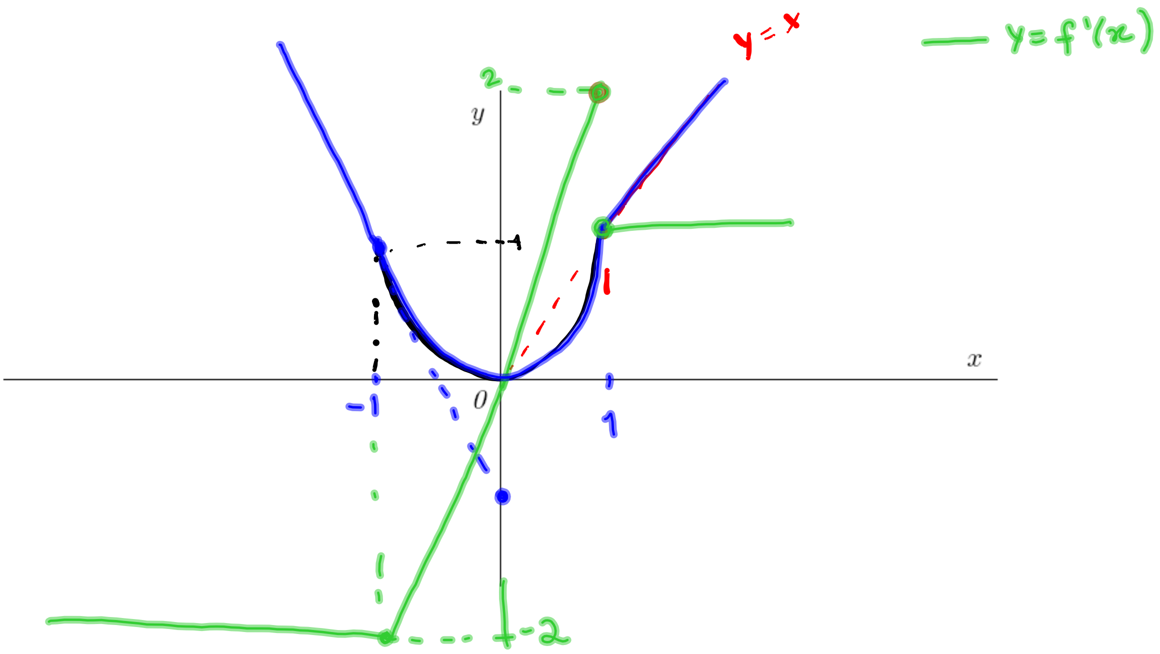
$x < 1 \quad f'_-(1) = 2$

$\qquad\qquad\qquad \neq \Rightarrow f'(1)$ DNE

$x > 1 \quad f'_+(1) = 1$

(b) *For what value(s) of* x *the function is not differentiable?* $\boxed{x = 1}$

(c) *Sketch the graph of* f *and* f' *on the same axis.*



EXAMPLE 6. *A ball is thrown into the air. Its position at time t is given by*

$$\vec{r}(t) = \langle 2t, 10t - t^2 \rangle .$$

(a) *Find the velocity of the ball at time $t = 2$.*

$$\vec{v}(t) = \vec{r}\,'(t) = \langle (2t)', (10t - t^2)' \rangle = \langle 2, 10 - 2t \rangle$$

$$\vec{v}(2) = \langle 2, 10 - 2 \cdot 2 \rangle = \langle 2, 6 \rangle$$

(b) *Find the speed of the ball at time $t = 2$.*

$$\text{speed} = |\vec{v}(2)| = |\langle 2, 6 \rangle| = \sqrt{2^2 + 6^2} = \sqrt{40} = 2\sqrt{10}$$

EXAMPLE 7. *Let*

→polynomial

$$f(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ ax+b & \text{if } x > 2 \end{cases}$$

Find the values of a and b that makes f differentiable everywhere.

We need continuity at $x=2$ and $f'_-(2) = f'_+(2)$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2) = 4$$

$x<2$; $x>2$

$$2^2 = \boxed{2a+b=4}$$

$x<2$; $x>2$

$$2x\Big|_{x=2} = a$$

$$\boxed{a=4}$$

$$b = 4-2a = 4-2\cdot 4 = -4$$

$$\boxed{b=-4}$$