Section 3.5: Chain Rule

Question: How to find the derivatives of the following functions:

$$y = (x^6 + 4x^2 + 12)^{15}; \qquad y = \sec(12x^2) + \tan^3(x) \qquad y = \sqrt[3]{4+x}$$

Review of Composite Functions:

composition

inner function

$$\boxed{[f \circ g](x) = f(g(x))}$$

outer function

If $f(x) = x^{15}$ and $g(x) = x^6 + 4x^2 + 12$ then

$$[f \circ g](x) = f(g(x)) = (x^6 + 4x^2 + 12)^{15}$$

"$[g(x)]^{15}$"

Conversely, if $[f \circ g](x) = \sec(12x^2)$ then $f(x) = \sec x$ and $g(x) = 12x^2$

The CHAIN RULE: *If the derivatives* $g'(x)$ *and* $f'(x)$ *both exist, and* $F = f \circ g$ *is the composite defined by*

$$F(x) = f(g(x))$$

then

$$F'(x) = f'(g(x))g'(x).$$

In Leibniz notation: If the derivatives of $y = f(u)$ *and* $u = g(x)$ *both exist then*

$$y = f(g(x)) = f(u)$$

is differentiable function of x *and*

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = y' \cdot u'$$

$y = f(x)$	$u(x)$ inner	$f(u)$ outer	$\frac{dy}{dx} = y' \cdot u'$
$y = (x^6 + 4x^2 + 12)^{15}$	$u = x^6 + 4x^2 + 12$ $u' = 6x^5 + 8x$	$y = u^{15}$ $y' = 15u^{14}$	$\frac{dy}{dx} = 15(x^6 + 4x^2 + 12)^{14} \cdot (6x^5 + 8x)$
$y = \sec(12x^2)$	$u = 12x^2$ $u' = 24x$	$y = \sec u$ $y' = \sec u \tan u$	$\frac{dy}{dx} = \underbrace{\sec(12x^2)\tan(12x^2)}_{y'} \cdot \underbrace{24x}_{u'}$
$y = \tan^3(x) = (\tan x)^3$	$u = \tan x$ $u' = \sec^2 x$	$y = u^3$ $y' = 3u^2$	$\frac{dy}{dx} = 3(\tan x)^2 \cdot \sec^2 x$
$y = \sqrt[3]{\underbrace{4+x}_{u}} = (4+x)^{1/3}$	$u = 4 + x$ $u' = 1$	$y = \sqrt[3]{u} = u^{1/3}$ $y' = \frac{1}{3} u^{-2/3}$	$\frac{dy}{dx} = \frac{1}{3}(4+x)^{-2/3}$
$y = [g(x)]^n$	$u = g(x)$ $u' = g'(x)$	$y = u^n$ $y' = nu^{n-1}$	$\frac{dy}{dx} = n[g(x)]^{n-1} g'(x)$ Generalized Power Rule

EXAMPLE 1. *Find the derivative:*

(a) $f'(x) = \left(\frac{1}{(x^3+5x^2+12)^{2012}}\right)' = (\underbrace{x^3+5x^2+12}_{u})^{-2012} = u^{-2012}$

$$f'(x) = n\,u^{n-1}\,u' = -2012\,u^{-2013}\,u'$$

$$f'(x) = -2012\,(x^3+5x^2+12)^{-2013}\,(3x^2+10x)$$

(b) $h(x) = x^8(3\sqrt{x}-11)^8 = \left(x(3\sqrt{x}-11)\right)^8 = (\underbrace{3x^{3/2}-11x}_{u})^8 = u^8$

$$h'(x) = 8u^7u' = 8(3x^{3/2}-11x)^7\left(3\cdot\tfrac{3}{2}x^{\frac{3}{2}-1}-11\right)$$

$$h'(x) = 8(3x^{3/2}-11x)^7\left(\tfrac{9}{2}x^{\frac{1}{2}}-11\right)$$

(c) $f(x) = \cos(\underbrace{5x}_{u}) + \cos^5 x = \cos(\underbrace{5x}_{u}) + (\underbrace{\cos x}_{u})^5$ Generalized Power Rule

$$f'(x) = -\sin(5x)\cdot 5 + 5(\cos x)^4(-\sin x)$$

(d) $f(x) = \sqrt{\underbrace{x^3+\sqrt{x^2+\sqrt{x}}}_{u}}$

Note $\boxed{(\sqrt{u})' = \frac{1}{2\sqrt{u}}}$

$$f'(u) = (\sqrt{u})'\cdot u' = \frac{1}{2\sqrt{x^3+\sqrt{x^2+\sqrt{x}}}}\cdot\left(3x^2+\frac{1}{2\sqrt{x^2+\sqrt{x}}}\cdot\left(2x+\frac{1}{2\sqrt{x}}\right)\right)$$

EXAMPLE 2. *Find F' and G' if*

$$F(x) = f(\overset{u}{\overbrace{\sin x}}), \qquad G(x) = \sin(\overset{u}{\overbrace{f(x)}}),$$

where $f(x)$ is a differentiable function.

$$F'(x) = f'(\sin x) \cdot (\sin x)' = f'(\sin x) \cdot \cos x$$

$$G'(x) = \cos(f(x)) \cdot f'(x)$$

<u>Note</u> $F(x) = [f \circ \sin](x)$, $G(x) = [\sin \circ f](x)$

EXAMPLE 3. *Let $f(x)$ and $g(x)$ be given differentiable functions satisfy the properties as shown in the table below:*

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	-5	8	3	12
3	1	2	-2	8

Suppose that $h = f \circ g$. Find $h'(1)$.

$$h(x) = f(g(x))$$

$$h'(x) = f'(g(x))\, g'(x)$$

$$h'(1) = f'(g(1))\, g'(1) = f'(3) \cdot 12$$

$$= 2 \cdot 12 = 24$$