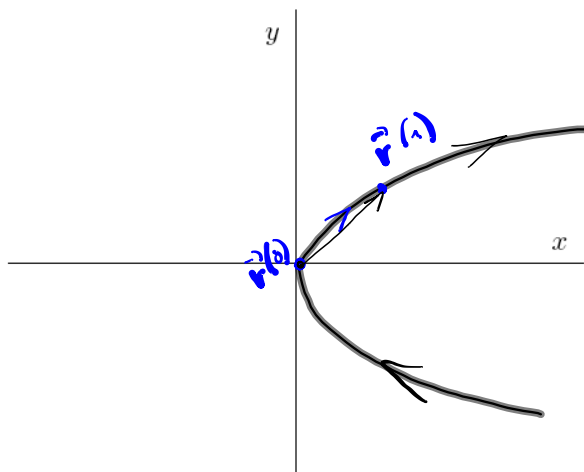


3.7: Derivatives of the vector functions (cf. 2.7)

EXAMPLE 1. Sketch the curve $\mathbf{r}(t)$ and indicate with arrow the direction in which t increases if

(a) $\mathbf{r}(t) = \langle t^2, t \rangle$



Eliminate parameter

$$\left. \begin{array}{l} x = t^2 \\ y = t \end{array} \right\} \Rightarrow x = y^2$$

$$\vec{r}(0) = \langle 0, 0 \rangle$$

$$\vec{r}(1) = \langle 1, 1 \rangle$$

(b) $\mathbf{r}(t) = \langle 2 \sin t, \cos t \rangle$

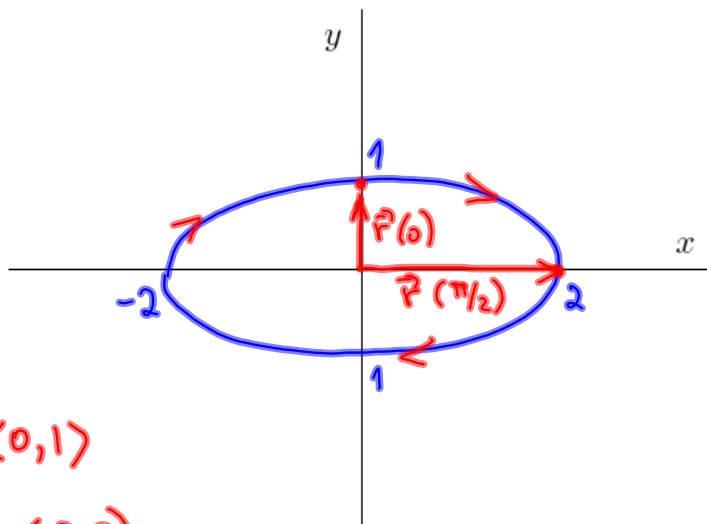
$$x = 2 \sin t \Rightarrow \sin t = \frac{x}{2}$$
$$y = \cos t$$

$$\sin^2 t + \cos^2 t = 1$$

$$\left(\frac{x}{2}\right)^2 + y^2 = 1$$

$$\frac{x^2}{4} + y^2 = 1$$

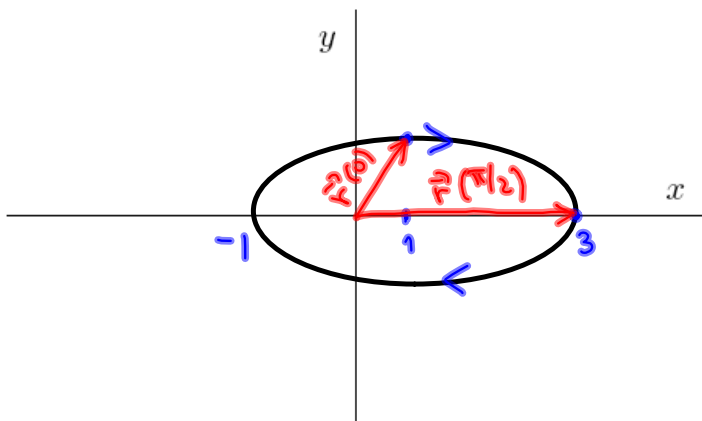
ellipse



$$\vec{r}(0) = \langle 0, 1 \rangle$$

$$\vec{r}\left(\frac{\pi}{2}\right) = \langle 2, 0 \rangle$$

(c) $\mathbf{r}(t) = \langle 1 + 2 \sin t, \cos t \rangle$



$$x = 1 + 2 \sin t \Rightarrow \sin t = \frac{x-1}{2}$$
$$y = \cos t$$

$$\cos^2 t + \sin^2 t = 1$$

$$y^2 + \left(\frac{x-1}{2}\right)^2 = 1$$

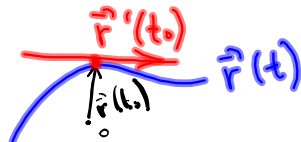
$$\frac{(x-1)^2}{4} + y^2 = 1$$

$$\left. \begin{array}{l} \vec{r}(0) = \langle 1, 1 \rangle \\ \vec{r}\left(\frac{\pi}{2}\right) = \langle 3, 0 \rangle \end{array} \right\} \text{ or use (b)}$$

cf. 2.7

Trajectory

DEFINITION 2. If $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ is a vector function, then



$$\text{velocity} = \mathbf{r}'(t) = \langle x'(t), y'(t) \rangle$$

if both $x'(t), y'(t)$ exist.

EXAMPLE 3. If $\mathbf{r}(t) = \langle t^2, \sqrt{t-5} \rangle$ find the domain of $\mathbf{r}(t)$ and $\mathbf{r}'(t)$.

$$\text{Domain } \mathbf{r}(t) = \{ \text{all } t, t \geq 5 \} \Rightarrow \{ t : t \geq 5 \} = \text{Domain } \mathbf{r}(t)$$

$$\vec{r}'(t) = \left\langle 2t, \frac{1}{2\sqrt{t-5}} \right\rangle \Rightarrow \text{Domain } \{ \vec{r}'(t) \} = \{ t : t > 5 \}$$

DEFINITION 4. If $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ is a vector function representing the position of a particle at time t , then

- instantaneous velocity at time t is $\mathbf{r}'(t) = \langle x'(t), y'(t) \rangle$
- instantaneous speed at time t is $|\mathbf{r}'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$

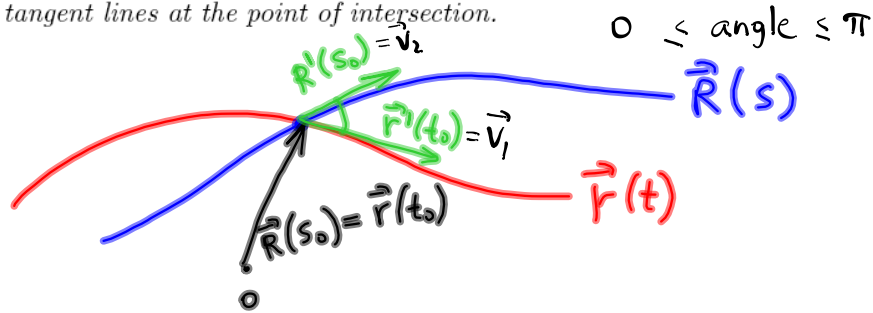
EXAMPLE 5. The vector function $\mathbf{r}(t) = \langle t, \sqrt{t^2 + 9} \rangle$ represents the position of a particle at time t . Find the velocity and speed of the particle at time $t = 4$.

$$\vec{r}'(t) = \left\langle 1, \frac{2t}{2\sqrt{t^2+9}} \right\rangle = \left\langle 1, \frac{t}{\sqrt{t^2+9}} \right\rangle$$

$$\vec{r}'(4) = \left\langle 1, \frac{4}{\sqrt{4^2+9}} \right\rangle = \left\langle 1, \frac{4}{5} \right\rangle \quad \text{velocity}$$

$$\text{speed} \quad |\vec{r}'(4)| = \sqrt{1 + \left(\frac{4}{5}\right)^2} = \sqrt{1 + \frac{16}{25}} = \frac{\sqrt{41}}{5}$$

DEFINITION 6. The angle between two intersecting curves (curvilinear angle) is defined to be the angle between the tangent lines at the point of intersection.



EXAMPLE 7. Given two curves traced by

$$r(t) = \langle 1+t, 3+t^2 \rangle, \quad R(s) = \langle 2-s, s^2 \rangle.$$

(a) At what point do the curves intersect?

$$\vec{r}(t) = \vec{R}(s)$$

$$\langle 1+t, 3+t^2 \rangle = \langle 2-s, s^2 \rangle$$

$$\begin{cases} 1+t = 2-s \Rightarrow t = 1-s \\ 3+t^2 = s^2 \end{cases}$$

$$3 + (1-s)^2 = s^2$$

$$3 + 1 - 2s + s^2 = s^2$$

$$2s = 4 \Rightarrow s = 2$$

$$t = 1 - s = 1 - 2 = -1$$

$$\vec{r}(-1) = \vec{R}(2) = \langle 1-1, 3+(-1)^2 \rangle = \langle 0, 4 \rangle$$

$$\boxed{(0, 4)}$$

(b) Find the angle between the curves.

Find tangent at $t = -1$ and $s = 2$

$$\vec{r}'(t) = \langle 1, 2t \rangle$$

$$\vec{R}'(s) = \langle -1, 2s \rangle$$

$$\vec{v}_1 = \vec{r}'(-1) = \langle 1, -2 \rangle$$

$$\vec{v}_2 = \vec{R}'(2) = \langle -1, 4 \rangle$$

$$\cos \angle \vec{v}_1, \vec{v}_2 = \frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1| \cdot |\vec{v}_2|} = \frac{\langle 1, -2 \rangle \cdot \langle -1, 4 \rangle}{\sqrt{1+4} \cdot \sqrt{1+16}} = -\frac{9}{\sqrt{85}}$$

$$\angle \vec{v}_1, \vec{v}_2 = \arccos\left(-\frac{9}{\sqrt{85}}\right) = \dots$$